Brake periodic orbits and linking in the calculus of variations

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Abstract

A combination of Galerkin's method, linking theory and Struwe's monotonicity method in the calculus of variations is used to study Hamiltonian systems in which the kinetic energy is given by a (not necessarily definite) quadratic form and the potential-energy functional may be bounded.

When the potential-energy functional is defined on a finite dimensional space, the existence of brake periodic orbits for almost all prescribed energies is established. An example of a Hamiltonian system which satisfies our hypotheses but has no brake periodic orbits with energy in an uncountable set of measure zero is given. Additional hypotheses, sufficient to ensure the existence of brake periodic orbits of all energies, are found.

Hypotheses are given which allow the potential to blow-up on the boundary of a bounded set. When the potential is even the existence of brake periodic orbits is achieved under conditions that allow the potential to have indefinite sign and so have unbounded sub-level sets.

For potential-energy functionals defined on an infinite dimensional Hilbert space, the existence of a brake periodic orbit is established for a countable set of positive energies. When the potential is even, existence is established for almost all energies.

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Contents

Chapter 1

Introduction

1.1 Outline of thesis

In this thesis we study Hamiltonian systems of the form

$$Su''(t) + \nabla V(u(t)) = 0, \tag{1.1a}$$

where V is defined on a Hilbert space H and S is a self-adjoint operator. The conserved energy has the form

$$\frac{1}{2}\langle Su'(t), u'(t)\rangle + V(u(t)) = h. \tag{1.1b}$$

By analogy with classical mechanics $\frac{1}{2}\langle Su', u'\rangle$ and $V(u)$ will be referred to as the kinetic and potential energies, respectively, and it is assumed that $V(0) = 0$. When the operator S is positive-definite the kinetic energy is said to be positive-definite. Similarly, when the operator S is indefinite the kinetic energy is said to be indefinite. A brake periodic orbit is a solution u of (1.1a) with the property that

$$u \text{ is non-constant, periodic and } u'(t_0) = u'(t_1) = 0, \tag{1.1c}$$

for some $t_0 \neq t_1$. We seek the existence of brake periodic orbits of (1.1a) with prescribed energy h. A classical system, in which $S = I$ (the identity) and the kinetic energy is positive-definite, is allowed, but $S = -I$ is excluded. Two cases of the Hilbert space H are discussed: $\mathbb{R}^{n+m}$ and l_2, where l_2 is the real separable Hilbert space of square summable sequences. When $H = \mathbb{R}^{n+m}$ the corresponding operator S has the form

$$S = \begin{pmatrix} I_{n\times n} & 0_{n\times m} \\ 0_{m\times n} & -I_{m\times m} \end{pmatrix}, \quad m \geq 0,\, n \geq 1. \tag{1.2}$$

Three cases are considered:

(C1) $V \in C^1(\mathbb{R}^{n+m}, [0,\infty))$,

(C2) $V \in C^1(\mathbb{R}^{n+m}, \mathbb{R})$ and V is even, and

(C3) $V \in C^1(\mathcal{O}, [0,\infty))$ where $\mathcal{O}$ is a bounded open subset of $\mathbb{R}^{n+m}$ containing the origin and $\lim_{x\to\partial\mathcal{O}} V(x) = \infty$.

When $H = l_2$ we suppose the corresponding operator S is an infinite dimensional symmetric matrix with the first $p \in \mathbb{N}$ entries equal to 1 and all others -1. Two cases are considered:

(C4) $V \in C^1(l_2, [0,\infty))$ and

(C5) $V \in C^1(l_2, \mathbb{R})$ and V is even.

In the above, case (Cn) is dealt with in Chapter n+2. In order to put the results and techniques of the thesis in context, we first discuss the classical case of positive-definite kinetic energy. We then turn to more recent studies involving indefinite kinetic energy. In both instances a motivating example is given in which the setting is finite dimensional.

In Section 1.4 we discuss an application of Struwe's monotonicity method which plays an important rôle in our existence theories; our use of the monotonicity method is new even in the simple case outlined in Section 1.4. Finally, in Section 1.5 we summarise the main results of the thesis.

1.2 Positive-definite kinetic energy

A motivating example

In the simple case when $m = 0$ and $n = 1$, (1.1a) represents the equation of motion of a bead, under the influence of gravity, that is threaded on a fixed frictionless wire. Equations (1.1b) and (1.1c) describe the case when the bead has prescribed energy h and oscillates back and forth along the wire with an unknown period. To see this, let $u(t)$ denote the distance at time t of the bead along the wire measured from a fixed point on the wire, and let $Y(u(t))$ denote the vertical coordinate of the bead. The Lagrangian is given by the kinetic energy minus the potential energy

$$\mathcal{L}(u, \dot{u}) = \frac{1}{2}\dot{u}^2 - gY(u),$$

where the bead is assumed to have unit mass and g is the magnitude of the constant gravitational field. The equation of motion of the bead is then given by

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\right) - \frac{\partial \mathcal{L}}{\partial u} = \ddot{u} + gY_u(u),$$

which is of the form of (1.1a) when $V(u) = gY(u)$. Moreover

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\dot{u} - \mathcal{L}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{u}^2 + gY(u)\right),$$

so that (1.1b) holds for some $h \in \mathbb{R}$.

Previous work with positive definite kinetic energy

The first result on the existence of solutions of (1.1) when S is positive-definite ($m = 0$ and $n \geq 1$) was due to Seifert [Sei48] who established existence under the assumption that $\{x \in \mathbb{R}^n : V(x) \leq h\}$ is homeomorphic to a closed ball with no critical points of V on its boundary. The result of Seifert has been extended in, for example, [Ben84], [GluZil83] and [Hay83]. Their common approach was to use the principle of Maupertuis (1698–1759) that states that solutions of (1.1) are geodesic lines, on a constant energy surface, relative to the Jacobi metric. Related is the following functional whose critical points admit a reparametrisation [Buf96] that transforms them into solutions of (1.1)

$$J_1(q) = \int_0^1 \sqrt{|q'(t)|^2\big(h - V(q(t))\big)}\,\mathrm{d}t.$$

Under the assumption of strict convexity of V, van Groesen [vanG88] established existence by considering the product functional

$$J_2(q) = \left(\int_0^1 |q'(t)|^2 \,\mathrm{d}t \right) \left(\int_0^1 h - V(q(t)) \,\mathrm{d}t \right).$$

Crucially, positive-valued critical points of J_2 correspond to solutions of (1.1). Benci and Giannoni [BenGia87] used the functional J_2 to prove the existence of a solution of (1.1) under the mild hypothesis that $\{x \in \mathbb{R}^n : V(x) < h\}$ is bounded and non-empty with no critical points of V on its boundary.

The functional J_2 is not necessarily bounded above or below so critical points do not arise from a straightforward maximisation or minimisation. Ambrosetti and Rabinowitz developed general minimax methods [AmbRab73, Rab78] to deal with such indefinite functionals, the simplest being the celebrated Mountain Pass Lemma (see for example [Wil96, Theorem 1.15], [AmbRab73], [Str00]).

Lemma 1.1 (Mountain Pass Lemma). *Suppose $J \in C^1(X, \mathbb{R})$ and there exists $q^* \in X$ and $r > 0$ such that $\|q^*\| > r$ with $J(q^*) \leq 0 = J(0)$ and $\inf_{q \in \partial B_r} J(q) =: \alpha > 0$. Then there exists a sequence $\{q_n\} \subset X$ and $c \in [\alpha, \infty)$ such that*

$$J(q_n) \to c \quad \textit{and} \quad \nabla J(q_n) \to 0 \quad \textit{as } n \to \infty.$$

Rabinowitz and Benci, while generalising the Mountain Pass Lemma, developed the notion of 'linking' [Rab78, RabBen79](see also Definition 2.2) which was successfully used in Benci's and Giannoni's existence proof [BenGia87].

1.3 Indefinite kinetic energy

Cases of indefinite S (in particular when $n = 1$), not all of which fit into one of our five cases, arise in important applications: the nonlinear theory of hydrodynamic waves [BonSmi76, BGT96, Tol81, Tol84], phase-front models [TerCha02] and periodic orbits of particles in rotating potentials [Buf96]. In the following we give an application that fits into case (C1).

Motivating example

The constrained nonlinear beam equation (see [HBT89], [HunEve99], [Hun00] and [HunWad91])

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + P\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - f'(y) = 0,$$

models a one-dimensional beam placed between elastic materials. The beam runs in the horizontal x direction and has vertical displacement y. The real parameter P represents the compressive load, and $-f'$ the restoring force provided by the materials. Configurations of the beam that are both periodic and buckled correspond to non-constant brake periodic orbits u of the system

$$-\begin{pmatrix} P & 1 \\ 1 & 0 \end{pmatrix} u'' + \nabla V(u) = 0 \quad \text{where} \quad u = \begin{pmatrix} y \\ y'' \end{pmatrix}$$

and the potential V is given by $V(u_1, u_2) = f(u_1) + \frac{1}{2}u_2^2$. The equation can, after a further change of variables (see Section 3.8), be put into case (C1) with $n = m = 1$. In [HBT89], functions f of the following form are considered

$$f(x) = x^k - c_1 x^2 \text{ for all } x \in \mathbb{R},$$

where $k = 3$ or $k = 4$ and $c_1 > 0$. If $k \geq 4$ is even, a change of origin is all that is needed for f to be covered by our theory. However, the case of odd k is excluded.

Previous work with indefinite kinetic energy

Early results on the existence of solutions of (1.1) when S is indefinite [HofTol84, Tol88], were obtained by non-variational methods, for $n = 1$ and arbitrary m, under the simple hypothesis that the sub-level set $\{x \in \mathbb{R}^{n+m} : V(x) < h\}$ is bounded and convex with no critical points of V on its boundary.

A maximisation approach was used by Buffoni [Buf96] for $n = 1$ and arbitrary m. He used the fact that, for a suitable admissible class of functions, critical points of the functional

$$J_4(q) = \int_0^1 \sqrt{-\langle Sq'(t), q'(t)\rangle}\sqrt{2(h - V(q(t)))}\,\mathrm{d}t$$

admit a reparametrisation which transform them into solutions of (1.1).

Buffoni and Giannoni [BufGia95] tackled the case of general $m \geq 0$ and $n \geq 1$ with the product functional

$$J_3(q) = \left(\int_0^1 \langle Sq'(t), q'(t)\rangle \,\mathrm{d}t\right)\left(\int_0^1 h - V(q(t))\,\mathrm{d}t\right)$$

and a generalisation of Benci and Giannoni's linking argument [BenGia87]. As before, positive-valued critical points of J_3 correspond to solutions of (1.1). When S is indefinite, Buffoni's and Giannoni's proof uses a Galerkin approach in an essential way; only when J_3 is somehow restricted to finite dimensions is there linking structure. Their hypotheses require the potential V to be increasing on rays through the origin and to grow at least quadratically.

1.4 Monotonicity method

In this thesis we use Struwe's parameter-dependence method, also called the monotonicity method [Str88, AmbStr89] (see also [Str00, Chapter II, Section 9]), to extend the linking-Galerkin approach of [BufGia95]. Jeanjean proved a monotonicity result in an abstract setting [Jea99] that was further refined in [JeaTol98]. In order to illustrate our use of Struwe's method we take a simple case and look at the functional J_2 in the setting of the Mountain Pass Lemma.

Suppose $V \in C^1(\mathbb{R}^n, [0,\infty))$ is even, $V(0) = 0$ and $\{x \in \mathbb{R}^n : V(x) < h\}$ is bounded for some $h > 0$. Let J_2 be defined on the following Hilbert space

$$X = \{q \in W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^n) : q(1-t) = q(1+t),\, q(-t) = -q(t)\, \forall t \in \mathbb{R}\},$$

where the norm is given by $||q||^2 = \int_0^1 |q'(t)|^2 \,\mathrm{d}t$. Then $J_2 \in C^1(X, \mathbb{R})$ and $J_2(0) = 0$. Since $V(0) = 0$ there exists $r > 0$ such that

$$J_2(q) \geq \frac{hr^2}{2} \quad \text{for all } q \in \partial B_r,$$

and since $\{x \in \mathbb{R}^n : V(x) < h\}$ is bounded, there exists $q^* \in X$ with $||q^*|| > r$ and

$J(q^*) \leq 0$. Therefore, by Lemma 1.1, there exists a sequence $\{q_n\} \subset X$ satisfying

$$J_2(q_n) = \|q_n\|^2 \int_0^1 h - V(q_n(t))\, \mathrm{d}t \to c \geq \frac{hr^2}{2} \quad \text{and} \quad \nabla J_2(q_n) \to 0 \quad \text{as} \quad n \to \infty.$$

The sequence $\{q_n\}$ is called a Palais-Smale sequence at level $c > 0$ and may be bounded or unbounded in X. If all such Palais-Smale sequences have strongly convergent subsequences then the existence of a positive-valued critical point is immediately established.

An elementary calculation shows that every *bounded* Palais-Smale sequence for the functional J_2 has a strongly convergent subsequence. Boundedness of Palais-Smale sequences can be deduced with additional assumptions on the potential V. For example, all positive level Palais-Smale sequences are bounded if there exists $p > 0$ such that $pV(x) \leq \langle \nabla V(x), x \rangle$ for all $x \in \mathbb{R}^n$.

When no additional assumptions are made about the potential V, the problem of getting bounded Palais-Smale sequences can be approached by the method of parameter dependence. Consider J_2 as a member of a one-parameter family of functionals $J_2(h, \cdot)$ with associated minimax level $c(h)$. The minimax principle can be arranged so that the function $h \mapsto c(h)$ is non-decreasing and so differentiable almost everywhere. For an energy h_0 for which $c'(h_0)$ exists, it can be shown by the techniques of Chapter 2, that there is a Palais-Smale sequence which is bounded in X by $c'(h_0) + 1$. So there is a solution of (1.1) for almost all $h > 0$ for which $\{x \in \mathbb{R}^n : V(x) < h\}$ is bounded.

1.5 Results of the thesis

In the first part of the thesis, the approach of Buffoni and Giannoni [BufGia95] is studied and extended to combine the linking-Galerkin approach with the parameter-dependence method to obtain the existence of solutions of (1.1) without any growth assumptions on V at infinity.

The existence results are a consequence of some abstract theory in Chapter 2 which develops that in [BufGia95]. What is new here is an adaptation of Struwe's monotonicity method [Str00, Chapter II, Section 9] (see also [AmbStr89, Str88]) to Galerkin's method. This yields *a priori bounds* that make possible the passage from the existence of finite-dimensional critical points to critical points in infinite dimensions. A simple application of Fatou's lemma is all that is needed to make Struwe's method uniform in a family of Galerkin approximations. This use of Fatou's lemma is similar to that in [Str00, pages 139–140]. Galerkin's method ensures that only very weak hypotheses on the behaviour of V are needed.

As in [BufGia95], Galerkin's method is essential to show that the functional has linking structure (Theorem 2.3); the finite-dimensional setting is also essential in the important observation of Lemma 3.8.

Chapter 2 and the majority of Chapter 3 have been published in [CriTol04].

1.5.1 Potentials defined on finite dimensions

In case (C1), that is when $V \in C^1(\mathbb{R}^{n+m}, [0, \infty))$ and $m \geq 0$, $n \geq 1$, the hypothesis $V \geq 0$ distinguishes the rôles of m and n. Because of (1.1b) and (1.1c), the total energy h of a solution belongs to the range of V. Let

$$0 < \mathcal{H}(V) := \liminf_{|x| \to \infty} V(x). \tag{1.3}$$

Then by (1.3) and the fact that V is C^1, the set

$$\mathcal{A}(V) = \{h \in [0, \mathcal{H}(V)) : V(x) = h,\ \nabla V(x) = 0 \text{ for some } x \in \mathbb{R}^{n+m}\} \tag{1.4}$$

is closed relative to $[0, \mathcal{H}(V))$ and has zero measure by Sard's theorem [Sar42]. When $h \in \mathcal{A}(V)$ there exists a constant solution u of (1.1a) where $u \equiv x \in \mathbb{R}^{n+m}$ for some x with $V(x) = h$.

The conclusions in Sections 3.1, 3.2 and 3.5 include the following:

(a) Suppose that $V \in C^1(\mathbb{R}^{n+m}, [0, \infty))$, $V(0) = 0$ and ∇V is bounded. Then there exists a solution of (1.1) for almost all $h \in (0, \mathcal{H}(V))$.

(b) If, in addition, $|x||\nabla V(x)| \to 0$ as $|x| \to \infty$, the periodic solutions in (a) are uniformly bounded, provided that h is bounded away from $\mathcal{H}(V)$, and there is a bounded even solution of (1.1a) and (1.1b) for all $h \in (0, \mathcal{H}(V))$.

(c) If, furthermore, $\langle x, \nabla V(x)\rangle > 0$ except when $x = 0$, then there exists a solution of (1.1) for all $h \in (0, \mathcal{H}(V))$.

The bounded even solution in (b) may coincide with the periodic solution in (a) when the latter exists; otherwise it may be a constant, or it may be non-constant but not periodic. In this context the following observation in Section 3.4 is relevant: for all H with $0 < H \leq \infty$ and for all relatively closed subsets A of $[0, H)$ with measure zero satisfying a gap-sum condition [BatNor96], there exists a C^2-function V satisfying our hypotheses, $\mathcal{H}(V) = H$, and there are no solutions of (1.1) when $h \in A$. This leads to an example for which there are no brake periodic orbits with energies in an uncountable set. Clearly such an example does not satisfy the hypothesis of (c) above.

In Section 3.6 a mild condition on the growth of V at infinity implies the existence of (periodic) solutions of (1.1) for all $h > 0$. This yields the main conclusion of [BufGia95] under somewhat weaker hypotheses.

When S is positive-definite, the equality (1.1b) implies $V(u(\cdot)) \leq h$; this simple *a priori* bound no longer holds if S is indefinite. In Section 3.1, maximum principle arguments are used with conditions on the potential to get *a priori* bounds on $u(\cdot)$ when S is indefinite. After the existence of brake periodic orbits have been established for almost all energies by linking-Galerkin arguments, the *a priori* bounds allow sufficient compactness to conclude, in Section 3.7, existence for a specified energy.

1.5.2 Evenness

In case (C2) we suppose $V \in C^1(\mathbb{R}^{n+m}, \mathbb{R})$ and $V(x) = V(-x)$ for all $x \in \mathbb{R}^{n+m}$. The evenness of the potential V allows us to deduce compactness of Palais-Smale sequences under weaker hypotheses. In particular, after careful consideration of the linking structure, it is apparent that V only has to be assumed positive on a subspace W of $\mathbb{R}^{n+m}$, where W has dimension $m+1$ and contains the negative eigenspace of S. (This also distinguishes the rôles of n and m.) We make the underlying assumption that

$$\mathcal{H}_W(V) := \liminf_{|P_W x| \to \infty} V(x) > 0,$$

where P_W is the orthogonal projection of $\mathbb{R}^{n+m}$ onto W. Our conclusions in Sections 4.2 and 4.5 include the following:

(d) Suppose $V \in C^1(\mathbb{R}^{n+m}, \mathbb{R})$ is even, $V(0) = 0 \leq V(x)$ for all $x \in W$ and $|x||\nabla V(x)| \leq 1 + V(x) + |x|$ for all $x \in \mathbb{R}^{n+m}$. Then there exists a solution of (1.1) for almost all $h \in (0, \mathcal{H}_W(V))$.

(e) If, in addition, $\langle P_W x, \nabla V(x) \rangle > 0$ for all $x \neq 0$, and $|V|$ is bounded, then there exists a solution of (1.1) for all $h \in (0, \mathcal{H}_W(V))$.

Even potentials are common in applications; for example, in the phase-front model [TerCha02] the corresponding potential function is even, of indefinite sign and the corresponding operator S is indefinite.

1.5.3 Potentials with blow-up

In case (C3) we suppose $V \in C^1(\mathcal{O}, [0, \infty))$ where $\mathcal{O}$ is a bounded open subset of $\mathbb{R}^{n+m}$ containing the origin and $\lim_{x \to \partial\mathcal{O}} V(x) = \infty$. The singularities of V at $\partial\mathcal{O}$ introduce an extra problem not present for the other cases: convergent Palais-Smale sequences may converge to a limit whose orbit intersects with $\partial\mathcal{O}$. Let Ω be the open set $\{q \in X : q(t) \in \mathcal{O}\}$. If the potential satisfies a 'strong force condition' of Gordon [Gor75], and $\{q_n\} \subset \Omega$ tends weakly and uniformly to $q \in X \setminus \Omega$, then

$$\int_0^1 V(q_n(t))\,\mathrm{d}t \to \infty \quad \text{as } n \to \infty.$$

Therefore, under the strong force condition, bounded positive-level Palais-Smale sequences of J_3 cannot converge to the singularity set $\partial\mathcal{O}$. Chapter 5 includes the following result.

(f) Let $V \in C^1(\mathcal{O}, [0, \infty))$ satisfy $V(0) = 0$ and $\lim_{x \to \partial\mathcal{O}} V(x) = \infty$. Suppose there exists an open neighbourhood $\mathcal{N}$ of $\partial\mathcal{O}$ and $U \in C^1(\mathcal{N} \cap \mathcal{O}, [0, \infty))$ satisfying

   (i) $\lim_{x \to \partial\mathcal{O}} U(x) = \infty$

   (ii) $V(x) \geq |\nabla U(x)|^2$ for all $x \in \mathcal{N} \cap \mathcal{O}$, and

   (iii) $\langle \nabla V(x), Sx \rangle \leq 2\langle \nabla V(x), x \rangle$ for all $x \in \mathcal{N} \cap \mathcal{O}$.

   Then there exists a solution of (1.1) for almost all $h > 0$.

1.5.4 Infinite dimensions

The systems in the first three cases are finite-dimensional. In case (C4) we suppose $V \in C^1(l_2, [0, \infty))$. The existence of solutions of (1.1) is established for a countable number of energies. The main idea is summarised as follows. Consider a sequence of potentials $V \circ P_n$, where P_n is an orthogonal projection of l_2 onto an n-dimensional subspace satisfying $\|(I - P_n)x\| \to 0$ as $n \to \infty$ for all $x \in l_2$. Adapt the linking-Galerkin-monotonicity method of Chapter 2 to get brake periodic orbits of the finite dimensional system in such a manner that bounds on the orbits are independent of n. Then introduce hypotheses on V to get a limit of the sequences that satisfies (1.1a) and (1.1c). Unless the potential is assumed to be even, the limiting energy is less than or equal to the energy of the approximating orbits. In cases (C4) and (C5), Sobolev spaces of Banach space valued functions are important; a summary of such spaces and the necessary integration theory is included in Appendices A and B. Chapter 7 discusses the special case of even infinite dimensional potentials. The conclusions of Theorem 6.6 and 7.2 include the following:

(g) Suppose $V \in C^1(l_2, [0, \infty))$ satisfies:

   (i) $V(0) = 0 < \liminf_{|x|_{l_2} \to \infty} V(x) =: \mathcal{H}_{l_2}(V)$,

   (ii) $|x||\nabla V(x)| \leq 1 + V(x) + |x|$ for all $x \in l_2$,

   (iii) if $x^k \rightharpoonup x$ weakly in l_2 as $k \to \infty$ then $\nabla V(x^k) \rightharpoonup \nabla V(x)$ weakly in l_2 as $k \to \infty$,

   (iv) if $x^k \rightharpoonup x$ weakly in l_2 as $k \to \infty$ then $\liminf_{k \to \infty} V(x^k) \geq V(x)$,

   (v) for each $i \in \mathbb{N}$, $\left(\nabla V\right)_i : l_2 \to \mathbb{R}$ is uniformly continuous on bounded sets, where $(x)_i$ denotes the i-th component of x with respect to the standard basis on l_2.

   Then for each $h > 0$ there is a solution of (1.1) with energy $h^* \in (0, h]$.

(h) If, in addition, the potential V is even, then there exists a solution of (1.1) for almost all $h \in (0, \mathcal{H}_{l_2}(V))$.

In the result (g) above, it is an open question as to whether there is existence for almost all $h \in (0, \mathcal{H}_{l_2}(V))$.

Chapter 2

Galerkin Theory of Critical Points

Let X be a separable Hilbert space and $X \neq Y \subset X$ a closed subspace so that $X = Y \oplus Z$ where $Z = Y^{\perp}$. For each $i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, let E_i be a finite-dimensional subspace of X such that $E_i = (E_i \cap Y) \oplus (E_i \cap Z)$, $E_1 \cap Z \neq \emptyset$, $E_i \subset E_{i+1}$ and $\cup_{i \in \mathbb{N}_0} E_i$ is dense in X. Let Ω be an open subset of X with $0 \in \Omega$ and let $\nabla u(x)$ denote the gradient at $x \in \Omega$, with respect to the inner product $\langle \cdot, \cdot \rangle$ in X, of a C^1-functional $u : \Omega \to \mathbb{R}$ and, if $x \in E_i \cap \Omega$, let $\nabla_i u(x) \in E_i$ denote the gradient of its restriction to E_i with respect to the same inner product.

Let $0 < h_1 < h_2 < \infty$, $e \in E_1 \cap Z$, $\|e\| = 1$ and let $\mathcal{V} : \Omega \to [0, \infty)$ be a C^1-functional such that $\mathcal{V}(0) = 0$. Define

$$\Lambda(h) = \{x \in \Omega : \mathcal{V}(x) < h\}$$

and suppose

(V1) $\sup\{\mu > 0 : y + \mu e \in \Lambda(h_2),\ y \in Y\} = M < \infty$;

(V2) $\Lambda_i(h_2) := \Lambda(h_2) \cap E_i$ is bounded for each $i \in \mathbb{N}_0$;

(V3) $\overline{\Lambda(h_2)} \subset \Omega$.

The set Ω is used to handle potentials which blow-up on the boundary of their domain of definition (see Chapter 5). When we are not considering such a case we let Ω equal X and then condition $(V3)$ is automatically satisfied.

Suppose that $\tau : X \times X \to \mathbb{R}$ is a continuous, symmetric, bilinear functional with

(T1) $\tau(y,z) = 0$ for all $(y,z) \in Y \times Z$;

(T2) $\tau(z,z) \geq c_0\|z\|^2 > 0$ for all $z \in Z \setminus \{0\}$;

(T3) $\tau(y,y) \leq 0$ for all $y \in Y$;

(T4) $\tau(x,\tilde{x}) = 0$ for all $(x,\tilde{x}) \in E_i \times E_i^{\perp}$.

Let

$$\mathcal{J}(h,x) = \tau(x,x)\big(h - \mathcal{V}(x)\big), \quad h \in [h_1,h_2], \quad x \in \Omega.$$

2.1 Critical points of $\mathcal{J}(h,\cdot)$ in E_i

Although it is possible to obtain the existence of critical points of $\mathcal{J}(h,\cdot)$ on $E_i \cap \Omega$ for all $i \in \mathbb{N}$ and for all $h \in [h_1,h_2]$, we could only establish part (i) of the following theorem for almost all h and for a subsequence $\{i_k\} \subset \mathbb{N}$. Part (i) is the essential step in the Galerkin approach to critical points in infinite dimensions when there are no growth hypotheses on $\mathcal{V}$. Theorem 3.12 gives an example of non-existence solutions of (1.1) for a set of energies which has zero measure. The following is therefore our main result.

Theorem 2.1. *For almost all $h_0 \in [h_1,h_2]$ there exists an increasing sequence $\{i_k\} \subset \mathbb{N}$, real numbers $\beta_2 > \beta_1 > 0$ and $C > 0$ such that restricted to $E_{i_k} \cap \Omega$ the functional $\mathcal{J}(h_0,\cdot)$ has a critical point $x_k \in E_{i_k} \cap \Omega$ with the following properties:*

(i) $0 < \tau(x_k,x_k) \leq C$ *(independent of k);*

(ii) $0 < \beta_1 \leq \mathcal{J}(h_0,x_k) \leq \beta_2 < \infty$ *(independent of k and h_0);*

(iii) $2\,\tau(x_k,x)\big(h_0 - \mathcal{V}(x_k)\big) = \tau(x_k,x_k)\langle \nabla_{i_k}\mathcal{V}(x_k),x\rangle$ *for all $x \in E_{i_k}$;*

(iv) $x_k \in \Lambda_{i_k}(h_0)$.

The proof is via a sequence of lemmas, culminating in Theorem 2.6. For future reference let $\tau_0 := \tau(e,e) > 0$. Define $J : [h_1,h_2] \times X \to \mathbb{R}$ by

$$J(h,x) = \begin{cases} \big(\tau(x,x)\big)^+ \big(h - \mathcal{V}(x)\big)^+, & x \in \Omega, \\ 0, & x \in X \setminus \Omega. \end{cases}$$

where $z^+ = \max\{0, z\}$, $z \in \mathbb{R}$. By (V3), $J(h,\cdot)$ is Lipschitz continuous for all $h \in [h_1, h_2]$. Note that $\mathcal{J}(h,x) = J(h,x)$ when $J(h,x) > 0$ and at all such points $J(h,\cdot)$ is continuously differentiable with

$$\langle \nabla J(h,x), \tilde{x}\rangle = 2\tau(x,\tilde{x})\big(h - \mathcal{V}(x)\big) - \tau(x,x)\langle \nabla \mathcal{V}(x), \tilde{x}\rangle$$

for all $\tilde{x} \in X$. By continuity and since $0 \in \Omega$ there exists $r > 0$ such that $\{x : \|x\| \leq r\} \subset \Omega$ and $\mathcal{V}(x) \leq h_1/2$ if $\|x\| \leq r$. Hence, by (T2),

$$\inf\big\{J(h,z) : \, z \in Z, \, \|z\| = r\big\} \geq c_0\, h_1\, r^2/2 =: \beta_1 \text{ for all } h \in [h_1, h_2]. \tag{2.1a}$$

In what follows we define, for each $\underline{h} \in [h_1, h_2)$, two linking sets (Lemma 2.3) and a corresponding minimax level (equation (2.4)) for the one-parameter family of functionals $J(h,\cdot)$ with $h \in [\underline{h}, h_2]$. In order to prove monotonicity of the minimax level (Lemma 2.4) it is necessary that the linking sets are fixed with respect to $h \in [\underline{h}, h_2]$. It is necessary to further restrict the range of the parameter h so that an important separation property holds (see equation (2.6)).

Let $\underline{h} \in [h_1, h_2)$ be arbitrary but fixed. Then, by hypotheses (V1) and (T), for any $h \in [\underline{h}, h_2]$,

$$\begin{aligned}
&\sup\big\{J(h, y+\mu e) : \mu > 0, \, y \in Y, \, y + \mu e \in \Lambda(\underline{h})\big\} = \\
&\sup\big\{\big(\tau(\mu e + y, \mu e + y)\big)^+ \big(h - \mathcal{V}(y+\mu e)\big)^+ : \mu > 0, \, y \in Y, \, y + \mu e \in \Lambda(\underline{h})\big\} \\
&\leq \tau_0 \sup\big\{\mu^2\big(h - \mathcal{V}(y+\mu e)\big)^+ : \mu > 0, \, y \in Y, \, y+\mu e \in \Lambda(\underline{h})\big\} \leq M^2 \tau_0\, h.
\end{aligned}$$

Hence for $h \in [\underline{h}, h_2]$,

$$\sup\big\{J(h, y+\mu e) : \mu > 0, \, y \in Y, \, y+\mu e \in \Lambda(\underline{h})\big\} \leq M^2\tau_0\, h_2 =: \beta_2 \tag{2.1b}$$

and, from (T3),

$$\sup\big\{J(h,y) : \, y \in Y \cap \Lambda(\underline{h})\big\} = 0 \text{ for all } h \in [\underline{h}, h_2]. \tag{2.1c}$$

Therefore, by (V1), (2.1b) and (2.1c)

$$\sup\big\{J(h, y+\mu e) : \mu \geq 0, \, y \in Y, \, y+\mu e \in \partial\Lambda(\underline{h})\big\} \leq M^2\tau_0\,(h - \underline{h}),$$

for all $h \in [\underline{h}, h_2)$. Let $\overline{h} \in (\underline{h}, h_2]$ then it follows that for all $h \in [\underline{h}, \overline{h}]$

$$\sup\left\{J(h, y + \mu e) : \mu \geq 0,\ y \in Y,\ y + \mu e \in \partial\Lambda(\underline{h})\right\} \leq \beta_1/2, \tag{2.1d}$$

where β_1 is defined in (2.1a), if

$$0 < \overline{h} - \underline{h} \leq c_0 \frac{h_1 r^2}{4M^2 \tau_0}, \quad [\underline{h}, \overline{h}] \subset [h_1, h_2], \tag{2.2}$$

where c_0 is given in (T2). Let

$$S = \left\{z \in Z : \|z\| = r\right\} \text{ and } Q(\underline{h}) = \left\{y + \mu e : \ y \in Y,\ \mu > 0\right\} \cap \Lambda(\underline{h}).$$

Note that $Q(\underline{h}) \subset \Lambda(h)$ for all $h \in [\underline{h}, \overline{h}]$. For each $i \in \mathbb{N}_0$ put

$$S_i = S \cap E_i \text{ and } Q_i(\underline{h}) = Q(\underline{h}) \cap E_i$$

and let $\partial_i Q(\underline{h})$ be the boundary of $Q_i(\underline{h})$ relative to $W_i := \operatorname{span}\{e,\ E_i \cap Y\}$. Note that

$$\partial_i Q(\underline{h}) \subset \left(Y \cap E_i \cap \overline{\Lambda(\underline{h})}\right) \cup \left(\partial\Lambda(\underline{h}) \cap \{y + \mu e : y \in Y \cap E_i,\ \mu \geq 0\}\right). \tag{2.3}$$

Let $\Gamma_i(\underline{h})$ denote the set of all continuous functions $\gamma : E_i \to E_i$ such that γ coincides with the identity on $\partial_i Q(\underline{h})$.

Definition 2.2. [Str00] *S_i and $\partial_i Q(\underline{h})$ link with respect to $\Gamma_i(\underline{h})$ if (a) $S_i \cap \partial_i Q(\underline{h}) = \emptyset$ and (b) $\gamma(Q_i(\underline{h})) \cap S_i \neq \emptyset$ for all $\gamma \in \Gamma_i(\underline{h})$.*

An important observation in Chapter 4 is that the following Lemma only requires $Q_i(\underline{h})$ to be a bounded open set.

Lemma 2.3. *S_i and $\partial_i Q(\underline{h})$ link with respect to $\Gamma_i(\underline{h})$ and, for all $h \in [\underline{h}, \overline{h}]$, $S_i \subset \Lambda_i(h)$.*

Proof. [Str00] (a) Let $x \in S_i$ then $x \in Z$, $\|x\| = r$, and $\mathcal{V}(x) \leq h_1/2 < \underline{h}$ by choice of r. Now $x \notin \partial_i Q(\underline{h})$ follows by (2.3) and $Z = Y^\perp$.
(b) Let P_i denote the orthogonal projection of E_i onto $Y \cap E_i$. It suffices to show that, for all $\gamma \in \Gamma_i(\underline{h})$, there exists $x \in Q_i(\underline{h})$ with $\|\gamma(x)\| = r$ and $P_i(\gamma(x)) = 0$.

Define $\tilde{\gamma} : [0,1] \times \overline{Q_i(\underline{h})} \to W_i$ by

$$\begin{aligned}\tilde{\gamma}(t, y+\mu e) &= tP_i(\gamma(y+\mu e)) + (1-t)P_i(y+\mu e) + ((1-t)\mu - tr)e \\ &\quad + t\|\gamma(y+\mu e) - P_i(\gamma(y+\mu e))\|e.\end{aligned}$$

Note that $\tilde{\gamma}$ is continuous and, from (V2), that $Q_i(\underline{h})$ is a bounded subset of W_i. For $y + \mu e \in \overline{Q_i(\underline{h})}$

$$\tilde{\gamma}(0, y+\mu e) = y + \mu e.$$

Hence $\tilde{\gamma}(0,\cdot)$ is the identity and so $\deg_B(Q_i(\underline{h}), \tilde{\gamma}(0,\cdot), 0) = 1$. In order to use homotopy invariance of degree [Llo78, Theorem 2.1.2(2)] it is required that $0 \notin \tilde{\gamma}([0,1], \partial_i Q(\underline{h}))$. Suppose that $y + \mu e \in \partial_i Q(\underline{h})$ and $t \in [0,1]$ with $\tilde{\gamma}(t, y+\mu e) = 0 \in W_i$. Then

$$\begin{aligned}0 &= t\|\gamma(y+\mu e) - P_i(\gamma(y+\mu e))\| + (1-t)\mu - tr \\ &= t\mu + (1-t)\mu - tr, \text{ since } \gamma(y+\mu e) = y+\mu e,\ \mu \geq 0 \text{ and } \|e\| = 1, \\ &= \mu - tr\end{aligned}$$

and $0 = ty + (1-t)y = y$. But $tre + 0 \notin \partial_i Q(\underline{h})$, a contradiction. Hence, by homotopy invariance of degree, $\deg_B(Q_i(\underline{h}), \tilde{\gamma}(1,\cdot), 0) = 1$.

Therefore $\tilde{\gamma}(1, y+\mu e) = 0$ has a solution in $Q_i(\underline{h})$. Therefore, $P_i(\gamma(y+\mu e)) = 0$ and $\|\gamma(y+\mu e) - P_i(\gamma(y+\mu e))\| = r$ imply $\|\gamma(y+\mu e)\| = r$. This proves that S_i and $\partial_i Q(\underline{h})$ link. To complete the proof of the lemma, note that the choice of r means that $S \subset \Lambda(h)$ for all $h \geq h_1$. ☐

Let

$$c_i(h, \underline{h}) = \inf_{\gamma \in \Gamma_i(\underline{h})} \max_{x \in Q_i(\underline{h})} J(h, \gamma(x)), \quad h \in [\underline{h}, \overline{h}]. \tag{2.4}$$

Since S_i and $\partial_i Q(\underline{h})$ link in E_i it follows from (2.1a)–(2.1d) that

$$0 < \beta_1 \leq \inf_{x \in S_i} J(h,x) \leq c_i(h, \underline{h}) \leq \sup_{x \in Q_i(\underline{h})} J(h,x) \leq \beta_2 < \infty \tag{2.5}$$

and, from (2.1c), (2.1d) and (2.3) that

$$\max_{x \in \partial_i Q(\underline{h})} J(h,x) \leq \beta_1/2, \quad h \in [\underline{h}, \overline{h}], \tag{2.6}$$

if $\overline{h} - \underline{h}$ satisfies (2.2).

Lemma 2.4. *For $i \in \mathbb{N}_0$, the function $h \mapsto c_i(h, \underline{h})$ is non-decreasing on $(\underline{h}, \overline{h})$.*

Proof. Fix $h, \widetilde{h} \in [\underline{h}, \overline{h}]$ with $\widetilde{h} > h$. Then for every $\gamma \in \Gamma_i(\underline{h})$ there exists, by (2.5) and (2.6), $\underline{x} \in Q_i(\underline{h})$ such that

$$J(h, \gamma(\underline{x})) = \max_{x \in Q_i(\underline{h})} J(h, \gamma(x)) \geq \beta_1 > 0.$$

Therefore $\tau(\gamma(\underline{x}), \gamma(\underline{x})) > 0$, $\mathcal{V}(\gamma(\underline{x})) < h$ and

$$\begin{aligned} \max_{x \in Q_i(\underline{h})} J(\widetilde{h}, \gamma(x)) \geq J(\widetilde{h}, \gamma(\underline{x})) &= J(h, \gamma(\underline{x})) + (\widetilde{h} - h)\tau(\gamma(\underline{x}), \gamma(\underline{x})) \\ &\geq \max_{x \in Q_i(\underline{h})} J(h, \gamma(x)) \geq c_i(h, \underline{h}). \end{aligned}$$

Taking the infimum of the left side over $\gamma \in \Gamma_i(\underline{h})$ yields the required result. □

Let $c_i'(h, \underline{h})$ denote $\big(\partial c_i / \partial h\big)(h, \underline{h})$. The monotonicity of $c_i(\cdot, \underline{h})$ means that its derivative exists and is non-negative almost everywhere and, although equality need not hold (because $c_i(\cdot, \underline{h})$ may not be absolutely continuous [Fri82, Lemma 2.14.4]),

$$\int_{\underline{h}}^{\overline{h}} c_i'(h, \underline{h})\, \mathrm{d}h \leq c_i(\overline{h}, \underline{h}) - c_i(\underline{h}, \underline{h}) \leq \beta_2 - \beta_1, \quad \text{by (2.5)}.$$

Let

$$\underline{c}(h, \underline{h}) := \liminf_{i \to \infty} c_i'(h, \underline{h}), \quad h \in [\underline{h}, \overline{h}].$$

Then, by Fatou's lemma,

$$\int_{\underline{h}}^{\overline{h}} \underline{c}(h, \underline{h})\, \mathrm{d}h \leq \beta_2 - \beta_1 < \infty.$$

So $\underline{c}(h, \underline{h})$ is finite for almost all $h \in [\underline{h}, \overline{h}]$. Now choose, and fix, $h_0 \in (\underline{h}, \overline{h})$ such that

$$0 \leq \alpha := \underline{c}(h_0, \underline{h}) < \infty. \tag{2.7}$$

By definition there exists an increasing sequence $\{i_k\}$ in $\mathbb{N}$ such that

$$c_{i_k}'(h_0, \underline{h}) \leq \alpha + 1/k \quad \text{for all } k \in \mathbb{N}. \tag{2.8}$$

For convenience with notation we will write c_k instead of c_{i_k}, E_k instead of E_{i_k} etc., from now on. In this notation let $\delta(k) > 0$ be such that

$$c_k(h, \underline{h}) - c_k(h_0, \underline{h}) \le (\alpha + 2/k)(h - h_0), \tag{2.9a}$$

$$c_k(h, \underline{h}) - 2(\alpha + 3/k)(h - h_0) > 3\beta_1/4, \tag{2.9b}$$

if $0 < h - h_0 \le \delta(k)$. The existence of $\delta(k)$ is guaranteed by (2.5) and (2.8). For any $h \in (h_0, \overline{h})$ there exists $\gamma \in \Gamma_k(\underline{h})$ such that

$$\max_{x \in Q_k(\underline{h})} J(h, \gamma(x)) \le c_k(h, \underline{h}) + (h - h_0)/k. \tag{2.10}$$

For any such γ and for all $x \in Q_k(\underline{h})$ for which

$$J(h_0, \gamma(x)) \ge c_k(h_0, \underline{h}) - (h - h_0)/k \tag{2.11a}$$

(2.9) and (2.10) imply that

$$\begin{aligned} \tau(\gamma(x), \gamma(x)) &= \frac{J(h, \gamma(x)) - J(h_0, \gamma(x))}{h - h_0} \\ &\le \frac{c_k(h, \underline{h}) - c_k(h_0, \underline{h}) + 2(h - h_0)/k}{h - h_0} \le \alpha + 4/k, \end{aligned}$$

for all h with $0 < h - h_0 < \delta(k)$. It follows that if $0 < h - h_0 < \delta(k)$, and inequalities (2.10) and (2.11a) hold, then

$$\tau(\gamma(x), \gamma(x)) \le \alpha + 4/k. \tag{2.11b}$$

Note also that when γ satisfies (2.10) then for all $x \in Q_k(\underline{h})$,

$$\begin{aligned} J(h_0, \gamma(x)) &= \left(\tau(\gamma(x), \gamma(x))\right)^+ \left(h_0 - \mathcal{V}(\gamma(x))\right)^+ \\ &\le J(h, \gamma(x)) \\ &\le c_k(h, \underline{h}) + (h - h_0)/k, \quad \text{since (2.10) holds,} \\ &= c_k(h_0, \underline{h}) + c_k(h, \underline{h}) - c_k(h_0, \underline{h}) + (h - h_0)/k \\ &\le c_k(h_0, \underline{h}) + (\alpha + 3/k)(h - h_0) \quad \text{if } 0 < h - h_0 < \delta(k). \end{aligned} \tag{2.11c}$$

Since $\Lambda_k(h_0)$ is bounded in E_k and τ is continuous, there exists $\sigma_k \in (0, \frac{1}{2}]$ such that

$$|\tau(x_1,x_1)-\tau(x_2,x_2)|\le 1/k \text{ when } x_1\in\Lambda_k(h_0) \text{ and } \|x_1-x_2\|<\sigma_k. \tag{2.12}$$

For $h\in(h_0,\overline{h})$ let

$$\begin{aligned}\mathcal{A}(h,k)=\{&x\in E_k\cap\Omega:\tau(x,x)\le\alpha+5/k,\\ &c_k(h_0,\underline{h})-(h-h_0)/k\le J(h_0,x)\le c_k(h_0,\underline{h})+(\alpha+3/k)(h-h_0)\big\}\end{aligned} \tag{2.13}$$

and define

$$d_k(h)=\inf\big\{\|\nabla_kJ(h_0,x)\|:x\in\mathcal{A}(h,k)\big\}. \tag{2.14}$$

The set $\mathcal{A}(h,k)$ is non-empty because of (2.11) and is a subset of $\Lambda(h_0)$ by (2.9b).

Since for each $k\in\mathbb{N}$ the mapping $J(h_0,\cdot):E_k\cap\Omega\to\mathbb{R}$ is continuously differentiable at $x\in E_k\cap\Omega$ whenever $J(h_0,x)>0$, there is associated with it (see [Str00, Chapter II, Section 3]) a pseudo-gradient vector field on $D_k:=\{x\in E_k\cap\Omega:J(h_0,x)>0,\nabla J(h_0,x)\ne 0\}$. That is, there exists a locally Lipschitz continuous mapping $\nu_k:D_k\to E_k$ satisfying

(i) $\|\nu_k(x)\|\le 2\min\{\|\nabla_kJ(h_0,x)\|,1\}$;

(ii) $\langle\nu_k(x),\nabla_kJ(h_0,x)\rangle\ge\min\{\|\nabla_kJ(h_0,x)\|,1\}\|\nabla_kJ(h_0,x)\|$.

Lemma 2.5.
$$\min\{d_k(h),1\}d_k(h)\le\frac{(\alpha+4/k)(h-h_0)}{\sigma_k}\ .$$

Proof. Suppose for a contradiction that the Lemma is false. For h with $0<h-h_0<\delta(k)$ let

$$\begin{aligned}A_h&=\big\{x\in\Omega:|J(h_0,x)-c_k(h_0,\underline{h})|\ge 2(\alpha+3/k)(h-h_0)\big\},\\ B_h&=\big\{x\in\Omega:c_k(h_0,\underline{h})-(h-h_0)/k\le J(h_0,x)\\ &\qquad\le c_k(h_0,\underline{h})+(\alpha+3/k)(h-h_0)\big\}.\end{aligned}$$

Note that $A_h\cap B_h=\emptyset$. Let N_1 and N_2 be open subsets of $\mathbb{R}^{n+m}$ such that

$$\{x\in E_k:\nabla_kJ(h_0,x)=0\}\subset\overline{N_1}\subset N_2 \text{ and } N_2\cap\mathcal{A}(h,k)=\emptyset.$$

The sets N_1 and N_2 exist by the assumption that the Lemma is false and the fact that $\mathcal{A}(h,k)$ is compact. Let

$$a(x) = \frac{\operatorname{dist}(x, A_h)}{\operatorname{dist}(x, A_h) + \operatorname{dist}(x, B_h)} \times \frac{\operatorname{dist}(x, N_1)}{\operatorname{dist}(x, N_1) + \operatorname{dist}(x, \mathbb{R}^{n+m} \setminus N_2)},$$

and define $W : E_k \to E_k$ by

$$W(x) = \begin{cases} -a(x)\,\nu_k(x), & x \in D_k, \\ 0, & x \notin D_k. \end{cases} \tag{2.15}$$

By hypothesis (V3), $\overline{A_h \cap E_k} \subset \Omega$, so W is locally Lipschitz continuous on E_k and $\|W(x)\| \le 2$ for all $x \in E_k$. Therefore the Cauchy problem

$$\dot{u}(t) = W(u(t)), \quad u(0) = x,$$

has a unique solution $u(t;x)$ defined for all t and $J(h_0, u(t;x))$ is a decreasing function of t for all x. Now define a homeomorphism U on E_k by

$$U(x) = u(\sigma_k; x), \quad x \in E_k,$$

where σ_k is defined in (2.12). If $x \in \partial_k Q(\underline{h})$ then (2.6) and (2.9) imply that $x \in A_h$ and so $U(x) = x$. This combined with the fact that U is a homeomorphism implies $U \circ \gamma \in \Gamma_k(\underline{h})$ for all $\gamma \in \Gamma_k(\underline{h})$. We consider in particular $U \circ \gamma$ when $\gamma \in \Gamma_k(\underline{h})$ satisfies the properties (2.11). Then there are two possibilities for $x \in Q_k(\underline{h})$.

In the first, $J(h_0, \gamma(x)) \le c_k(h_0, \underline{h}) - (h - h_0)/k$ and consequently

$$J(h_0, U \circ \gamma(x)) \le c_k(h_0, \underline{h}) - (h - h_0),$$

because $J(h_0, u(t; \gamma(x)))$ is decreasing in t.

In the second (2.11a), and hence (2.11b) and (2.11c), hold. For convenience let $\widehat{x} = \gamma(x)$ in this case. Then either

$$J(h_0, U(\widehat{x})) < c_k(h_0, \underline{h}) - (h - h_0)/k,$$

or, since $J(h_0, u(t;\widehat{x}))$ is decreasing,

$$J(h_0, u(t;\widehat{x})) \in [c_k(h_0, \underline{h}) - (h - h_0)/k, c_k(h_0, \underline{h}) + (\alpha + 3/k)(h - h_0)]$$

for all $t \in (0, \sigma_k)$. Suppose we are in the latter case, that is $u(t;\widehat{x}) \in B_h$ for all $t \in (0, \sigma_k)$. Then, by definition of J and (2.9b), $u(t;\widehat{x}) \in \Lambda(h_0)$ for all $t \in (0, \sigma_k)$. Because of the choice of σ_k and the fact that $\|W\| \leq 2$, it follows from (2.11b) that

$$\tau\big(u(t;\widehat{x}), u(t;\widehat{x})\big) \leq \alpha + 5/k \text{ for all } t \in (0, \sigma_k).$$

Hence $u(t;\widehat{x}) \in \mathcal{A}(h,k)$ for all $t \in (0, \sigma_k)$ and so, by definition,

$$\begin{aligned} J(h_0, U(\widehat{x})) &= J(h_0, \widehat{x}) - \int_0^{\sigma_k} \langle \nu_k(u(t;\widehat{x}), \nabla_k J(h_0, u(t;\widehat{x}))\rangle \, \mathrm{d}t \\ &\leq J(h_0, \widehat{x}) - \int_0^{\sigma_k} \min\{\|\nabla_k J(h_0, u(t;\widehat{x}))\|, 1\}\|\nabla_k J(h_0, u(t;\widehat{x}))\| \, \mathrm{d}t \\ &\leq J(h_0, \widehat{x}) - \sigma_k \min\{d_k(h), 1\} d_k(h), \end{aligned}$$

by the definition of the pseudo-gradient and the fact that $t \mapsto t\min\{t, 1\}$ is increasing. Since we are supposing that the conclusion of the Lemma is false

$$\begin{aligned} J(h_0, U(\widehat{x})) &< c_k(h_0, \underline{h}) + (\alpha + 3/k)(h - h_0) - \sigma_k\Big(\frac{(\alpha + 4/k)(h - h_0)}{\sigma_k}\Big) \\ &= c_k(h_0, \underline{h}) - (h - h_0)/k. \end{aligned}$$

This shows that

$$\max_{x \in Q_k(\underline{h})} J(h_0, U \circ \gamma(x)) \leq c_k(h_0, \underline{h}) - (h - h_0)/k.$$

Since $U \circ \gamma \in \Gamma_k(\underline{h})$, this contradicts the definition of $c_k(h_0, \underline{h})$, and the lemma is proven. □

Theorem 2.6. *For all $k \in \mathbb{N}$ there exists a critical point x_k of $\mathcal{J}(h_0, \cdot)$ in $E_k \cap \Omega$*

with

$$0 < \beta_1 \le c_k(h_0, \underline{h}) = \mathcal{J}(h_0, x_k) \le \beta_2; \tag{2.16a}$$

$$0 < \tau(x_k, x_k) \le \alpha + 5/k \quad (\alpha \textit{ independent of } k); \tag{2.16b}$$

$$2\,\tau(x_k, x)\big(h_0 - \mathcal{V}(x_k)\big) = \tau(x_k, x_k)\langle \nabla_k \mathcal{V}(x_k), x\rangle \textit{ for all } x \in E_k. \tag{2.16c}$$

Proof. The existence of a critical point of $\mathcal{J}(h_0, \cdot)$ on $E_k \cap \Omega$ with these properties is an immediate consequence of the preceding lemma, the fact that $\mathcal{J}(h, x) = J(h, x)$ whenever $J(h, x) > 0$, $\mathcal{J}(h_0, \cdot) \in C^1(\Omega, \mathbb{R})$, the compactness of $\overline{\Lambda(\overline{h})}$ in the finite-dimensional space E_k and the estimates (2.5). □

Proof of Theorem 2.1. Since h_0 was chosen from a set of full measure in $[\underline{h}, \overline{h}]$, and since the right side of (2.2) is independent of $\underline{h}$, an arbitrary point of $[h_1, h_2)$, Theorem 2.1 follows from Theorem 2.6 with $C = \alpha + 5$. □

2.2 Critical points in X

To extend the existence of critical points to the infinite dimensional setting we introduce the following hypotheses. In what follows β_1 and β_2 are defined, in terms of $\mathcal{V}$, τ, h_1 and h_2, by (2.1a) and (2.1b), and for almost all h the constant C, which depends on h, is given by Theorem 2.1. However, when it comes to verifying the following hypotheses in the context of brake periodic orbits, any $\beta_2 \ge \beta_1 > 0$ and $C > 0$ will do. We suppose that, for almost all $h \in [h_1, h_2]$,

(H1) $\cup_{i\in\mathbb{N}}\{x \in \Lambda_i(h) : \nabla_i \mathcal{J}(h, x) = 0,\ \beta_1 \le \mathcal{J}(h, x) \le \beta_2,\ 0 < \tau(x, x) \le C\}$ is a bounded set, denoted by $R(h) \subset X$, when $\beta_2 > \beta_1 > 0$ and $C > 0$;

(H2) $\langle N(x), \tilde{x}\rangle = \tau(x, x)\langle \nabla \mathcal{V}(x), \tilde{x}\rangle$ for all $x \in \Omega$ and $\tilde{x} \in X$ defines a compact operator N on Ω;

(H3) if $\{x_i\} \subset R(h)$ and $\nabla \mathcal{J}(h, x_i) \to 0$ as $i \to \infty$ in X^*, then $\{x_i\}$ has subsequence which is strongly convergent in X.

Theorem 2.7. *Let $\mathcal{V}$ and τ satisfy conditions (V) and (T) of Theorem 2.1 and suppose (H) holds. Then there exists $\beta_1, \beta_2 \in \mathbb{R}$ such that for almost all $h \in [h_1, h_2]$ there exists $x \in \Lambda(h)$ with $\nabla \mathcal{J}(h, x) = 0$, $0 < \beta_1 \le \mathcal{J}(h, x) \le \beta_2$ and $\tau(x, x) \le \alpha$ (where α is given by (2.7)).*

Proof. Since $\mathcal{V}$ and τ satisfy (V) and (T), all the conditions of Theorem 2.1 are satisfied. Therefore, for almost all $h \in [h_1, h_2]$, there is an increasing subsequence $\{i_k\} \subset \mathbb{N}$ and $0 < \beta_1 < \beta_2$ such that, for each k, there exists $x_k \in R(h)$ with $\nabla_{i_k} \mathcal{J}(h, x_k) = 0$ and $\tau(x_k, x_k) \leq \alpha + 5/k$. For almost all $h \in [h_1, h_2]$ the hypotheses (H1)-(H3) also hold with $C = \alpha + 5$. Therefore, without loss of generality, we may assume that $\{x_k\}$ is bounded in Ω and, as N is compact, that $N(x_k) \to w$ as $k \to \infty$, for some $w \in X$. Note that

$$\Pi_{i_k} \nabla \mathcal{J}(h, x_k) = \nabla_{i_k} \mathcal{J}(h, x_k) = 0,$$

where Π_{i_k} is the orthogonal projection from X onto E_{i_k}. If $\tilde{x} \in E_k^{\perp}$ then $\tau(x_k, \tilde{x}) = 0$ by (T4), and

$$\langle \nabla \mathcal{J}(h, x_k), \tilde{x} \rangle + \langle N(x_k), \tilde{x} \rangle = 2\tau(x_k, \tilde{x})(h - \mathcal{V}(x_k)) = 0.$$

Therefore $\nabla \mathcal{J}(h, x_k) + N(x_k) \in E_k$. Hence

$$\begin{aligned} \|\nabla \mathcal{J}(h, x_k)\| &= \|\Pi_{i_k} \nabla \mathcal{J}(h, x_k) - (I - \Pi_{i_k}) N(x_k)\| = \|(I - \Pi_{i_k}) N(x_k)\| \\ &\leq \|(I - \Pi_{i_k}) w\| + \|N(x_k) - w\| \to 0, \end{aligned}$$

as $k \to \infty$. So, by (H3), $\{x_k\}$ has a strongly convergent subsequence with limit $x \in \overline{\Lambda(h)}$ satisfying $\tau(x, x) \leq \alpha$. By (V3), $x \in \Omega$. Since $\mathcal{J}(h, \cdot)$ is a C^1- functional, $\nabla \mathcal{J}(h, x) = 0$ and $0 < \beta_1 \leq \mathcal{J}(h, x) \leq \beta_2$. If $x \in \partial \Lambda(h)$ then $\mathcal{J}(h, x) = 0$ so $x \in \Lambda(h)$. □

In certain circumstances, which in applications amounts to a growth condition on $\mathcal{V}$, it is possible to find critical points for all $h \in [h_1, h_2]$.

(H4) When $0 < \beta_1 < \beta_2$, the set

$$\mathcal{R} := \cup_{h \in [h_1, h_2]} \{x \in \Lambda(h) : \nabla \mathcal{J}(h, x) = 0, \beta_1 \leq \mathcal{J}(h, x) \leq \beta_2\}$$

is relatively compact in X.

Theorem 2.8. *Suppose (T), (V) and (H1)-(H4) hold. Then for all $h \in [h_1, h_2]$ there exists $x \in \Lambda(h)$ such that $\nabla \mathcal{J}(h, x) = 0$ and $\beta_1 \leq \mathcal{J}(h, x) \leq \beta_2$.*

Proof. Fix $h \in [h_1, h_2)$ and let $0 < h_1 \leq \underline{h} \leq h < \overline{h} < h_2$ satisfy (2.2). Then by Theorem 2.7 there exist sequences $\{h_j\} \subset (\underline{h}, \overline{h})$, $\{x_j\} \subset X$ and $\beta_1, \beta_2 \in \mathbb{R}$ (independent of j) such that; $h_j \to h$ as $j \to \infty$, $x_j \in \Lambda(h_j)$, $0 < \beta_1 \leq \mathcal{J}(h_j, x_j) \leq \beta_2$ and $\nabla\mathcal{J}(h_j, x_j) = 0$. In other words, $\{x_j\} \subset \mathcal{R}$. Therefore $\{x_j\}$ has a strongly convergent subsequence. Since $\mathcal{J}$ is a C^1-functional, $\mathcal{J}(h_j, x_j) \to \mathcal{J}(h, x)$ and $\nabla\mathcal{J}(h_j, x_j) \to \nabla\mathcal{J}(h, x)$ as $j \to \infty$ and the result follows. $\square$

Chapter 3

Brake Periodic Orbits

We turn now to a concrete realisation of the space X and the functional $\mathcal{J}$, the critical points of which yield solutions of (1.1). Let

$$X = \{q \in W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^{n+m}) : q(t) = q(2+t) \text{ and } q(-t) = q(t) \, \forall t \in \mathbb{R}\},$$

which is a Hilbert space when endowed with inner product

$$\langle q_1, q_2 \rangle = \int_0^1 \langle q_1(t), q_2(t) \rangle + \langle q_1'(t), q_2'(t) \rangle \, \mathrm{d}t.$$

Define $\mathcal{J} : \mathbb{R} \times X \to \mathbb{R}$ by

$$\mathcal{J}(h, q) = \tau(q, q)(h - \mathcal{V}(q)) \tag{3.1}$$

where $\mathcal{V} : X \to \mathbb{R}$ and $\tau : X \times X \to \mathbb{R}$ are given by

$$\mathcal{V}(q) = \int_0^1 V(q(t)) \, \mathrm{d}t \quad \text{and} \quad \tau(q_1, q_2) = \int_0^1 \langle S q_1'(t), q_2'(t) \rangle \, \mathrm{d}t,$$

where the operator S is defined in (1.2). Clearly τ is a continuous symmetric bilinear functional on X. Therefore, by Theorem C.1, $\mathcal{J}$ is C^1 on $\mathbb{R} \times X$ since V is C^1.

Lemma 3.1. *For $h > 0$, critical points q of $\mathcal{J}(h, \cdot)$ with $\mathcal{J}(h, q) > 0$ can be* *re-scaled in t to give solutions of* (1.1) *with $t_1 = 0$ and*

$$t_0^2 = \frac{\mathcal{J}(h,q)}{2\big(h - \int_0^1 V(q(t))\,\mathrm{dt}\big)^2} > 0.$$

Proof. For such a critical point, $q \in C^2(\mathbb{R}, \mathbb{R}^{n+m})$,

$$Sq''(t) + t_0^2 \nabla V(q(t)) = 0, \quad q'(0) = q'(1) = 0, \text{ and}$$

$$\frac{1}{2}\langle Sq'(t), q'(t)\rangle + t_0^2 V(q(t)) = \int_0^1 \frac{1}{2}\langle Sq'(t), q'(t)\rangle + t_0^2 V(q(t))\,\mathrm{d}t = t_0^2 h.$$

Therefore $u(t) = q(t/t_0)$ is a solution of (1.1). □

3.1 A priori bounds

Before going further we prove two elementary results on the boundedness of solutions of (1.1), which are useful when V is bounded or of slow growth and significant because the Hamiltonian is not coercive.

Theorem 3.2. *Suppose that*

$$2(V(x) - h) - \langle Sx, \nabla V(x)\rangle > 0 \textit{ for all } x \textit{ with } |x| > R.$$

Then $|u(t)| \le \max\{R, |u(t_0)|, |u(t_1)|\}$ *if u satisfies* (1.1).

Proof. Suppose that u satisfies (1.1) and let $f(t) = |u(t)|^2/2$. Then $f(t_0) = |u(t_0)|^2/2$, $f(t_1) = |u(t_1)|^2/2$ and, if the result is false, there exists a maximiser $t^* \in (t_0, t_1)$ of f at which $f''(t^*) \le 0$ and $|u(t^*)| > R$. Therefore

$$\begin{aligned}
0 &\ge f''(t^*) = |u'(t^*)|^2 + \langle u(t^*), u''(t^*)\rangle \\
&= |u'(t^*)|^2 - \langle u(t^*), S\nabla V(u(t^*))\rangle \text{ by (1.1a)} \\
&\ge -\langle Su'(t^*), u'(t^*)\rangle - \langle u(t^*), S\nabla V(u(t^*))\rangle \text{ by definition of } S \\
&= 2(V(u(t^*)) - h) - \langle u(t^*), S\nabla V(u(t^*))\rangle \text{ by (1.1b)} \\
&> 0
\end{aligned}$$

because of our hypothesis. This contradiction completes the proof. □

Remark 3.3. *The hypothesis of this theorem is commonly satisfied when V is bounded or grows sub-linearly at infinity. For example, when V is bounded and* $\lim_{|x|\to\infty}|x|\,|\nabla V(x)|=0$, *solutions u of* (1.1) *lie in a ball with radius independent of h when $\mathcal{H}(V)-h\geq\delta$ for some $\delta>0$.*

The next *a priori* bound on the range of solutions of (1.1) also uses a maximum principle. This time an assumption on the Hessian of V is made necessitating the assumption of higher differentiability of V. A potential V is given, with slow growth, that satisfies the hypothesis of the Lemma.

Lemma 3.4. *Suppose there exists $r>0$ such that whenever $x\in\mathbb{R}^{n+m}$ and $V(x)\geq h+r$,*

$$\langle S\nabla V(x),\nabla V(x)\rangle<\langle\nabla^2V(x)y,y\rangle \textit{ for all } y\in\mathbb{R}^{n+m} \textit{ with } \langle Sy,y\rangle<-2r.$$

Then if u is a solution of (1.1) with energy h,

$$V(u(t))\leq h+r,\ \textit{ for all } t\in\mathbb{R}.$$

Proof. Let u be a solution of (1.1) with energy h, and define $f(t)=h-V(u(t))$ for all $t\in\mathbb{R}$. Since $u'(0)=0=u'(t_0)$, $f(0)=0=f(t_0)$ by (1.1b).

Suppose the conclusion of the Lemma is false. Then there exists a global minimiser $t^*\in(0,t_0)$ of f with $f''(t^*)\geq 0$ and $f(t^*)<-r$. Hence, by (1.1b),

$$\frac{1}{2}\langle Su'(t^*),u'(t^*)\rangle=h-V(u(t^*))<-r. \tag{3.2}$$

Then, $f'(t^*)=-\langle\nabla V(u(t^*)),u'(t^*)\rangle$ and, by (3.2) and (1.1a),

$$0\leq f''(t^*)=-\langle\nabla^2V(u(t^*))u'(t^*),u'(t^*)\rangle+\langle S\nabla V(u(t^*)),\nabla V(u(t^*))\rangle<0.$$

This palpable contradiction completes the proof. □

Corollary 3.5. *Suppose there exists $r>0$ such that whenever $x\in\mathbb{R}^{n+m}$ and $V(x)\geq h+r$,*

$$|\nabla V(x)|^2<2rm(x),$$

where $m(x):=inf_{|y|=1}\langle\nabla^2V(x)y,y\rangle$ is the least eigenvalue of $\nabla^2V(x)$. Then if u

is a solution of (1.1) with energy h,

$$V(u(t)) \leq h + r, \ \textit{for all } t \in \mathbb{R}.$$

□

Remark 3.6. *If the condition in Lemma 3.4 is satisfied for some $h^*, r > 0$, then all solutions of (1.1) with energy $h \leq h^*$ satisfy*

$$V(u(t)) \leq h^* + r, \quad \textit{for all } t \in \mathbb{R}.$$

This means that if there is a sequence of solutions, denoted $\{u_n\}$, of (1.1) with energies $h_n \in (0, h^*)$ and $0 < h^* < \mathcal{H}(V)$ then $\{u_n\}$ is bounded in L^∞.

In practice, when verifying the condition, it is sufficient to consider $2r < |y|^2$, since this is implied by $\langle Sy, y\rangle < -2r$.

The next example shows that the condition in Lemma 3.4 can be satisfied for some V with slow growth.

Let $V(x) = g(|x|)$ for all $x \in \mathbb{R}^{n+m}$ where $g : \mathbb{R} \to [0, \infty)$ is continuously differentiable and such that whenever $g(s) \geq h + r$ with $s > 0$,

$$\text{(i) } g''(s) + g'(s)s^{-1} \leq 0 \ \text{ and (ii) } \ 0 < g'(s)^2 \leq 2r(g''(s) + 2g'(s)s^{-1}).$$

Such a condition holds if, for example, $r \geq 1/2$ and $g(s) = \log s$ whenever $s \geq \exp(h + r)$. Then for all $x, y, z \in \mathbb{R}^{n+m}$ with $x \neq 0$, $\nabla V(x) = g'(|x|)|x|^{-1}x$ and

$$\langle \nabla^2 V(x)y, z\rangle = \frac{g''(|x|)}{|x|^2}\langle x, y\rangle\langle x, z\rangle + \frac{g'(|x|)}{|x|}\left(\langle z, y\rangle + \frac{\langle x, y\rangle\langle x, z\rangle}{|x|^2}\right).$$

Suppose that $|x| \geq \exp(h + r)$ and $|y|^2 > 2r$. Then by Schwarz's inequality and

(i)

$$\begin{aligned}
\langle \nabla^2 V(x)y, y\rangle &= \frac{g'(|x|)}{|x|}|y|^2 + \left(g''(|x|) + \frac{g'(|x|)}{|x|}\right)\frac{\langle x, y\rangle^2}{|x|^2} \\
&\geq \frac{g'(|x|)}{|x|}|y|^2 + \left(g''(|x|) + \frac{g'(|x|)}{|x|}\right)|y|^2 \\
&= \left(2\frac{g'(|x|)}{|x|} + g''(|x|)\right)|y|^2,
\end{aligned}$$

and hence by (ii) and the fact that $|y|^2 > 2r$,

$$\begin{aligned}
&> \left(2\frac{g'(|x|)}{|x|} + g''(|x|)\right)2r \\
&\geq g'(|x|)^2 \geq g'(|x|)^2\frac{\langle Sx, x\rangle}{|x|^2} \\
&= \langle S\nabla V(x), \nabla V(x)\rangle.
\end{aligned}$$

3.2 Existence for almost all energies

To apply Theorems 2.7 and 2.8 to $\mathcal{J}$ let

$$E_i = \operatorname{span}\{e_{j,k} : 0 \leq k \leq i,\ 1 \leq j \leq m+n\}, \quad i \in \mathbb{N}_0,$$

where, for $1 \leq j \leq n+m$ and $k \in \mathbb{N}_0$,

$$e_{j,k}(t) := (0, \ldots, 0, \underbrace{\cos k\pi t}_{j\text{-th coeff.}}, 0, \ldots, 0) \in \mathbb{R}^{n+m}.$$

For each $q \in X$ write $q(t) = (z(t), y(t))$ where $z(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^m$. Let

$$\begin{aligned}
Y &= \{(c, y) \in X : c \in \mathbb{R}^n\}, \\
Z &= Y^\perp = \left\{(z, 0) \in X : \int_0^1 z(t)\,\mathrm{d}t = 0\right\}.
\end{aligned}$$

Note that $\|e_{1,1}\| = 1$ and put $(e, 0) = e_{1,1} \in E_1 \cap Z$.

Lemma 3.7. *If $q \in E_k \setminus \{0\}$, then $q(t) = 0$ at most at finitely many points $t \in [0, 1]$.*

Proof. Since every element of E_i is real-analytic, the result is immediate. $\square$

Lemma 3.8. *Suppose $\{q_j\}$ is a sequence in E_i with $\|q_j\| \to \infty$ as $j \to \infty$. Then for all $\gamma \in (0,1)$ there exists a set $U \subset [0,1]$ with $\operatorname{meas} U > \gamma$ and a subsequence $\{q_{j_k}\}$ (independent of γ) such that $|q_{j_k}(t)| \to \infty$ uniformly for $t \in U$.*

Proof. All elements of E_i are continuous functions and since E_i is finite dimensional, all norms on E_i are equivalent. Let $\|\cdot\|$ denote the usual L_∞-norm on E_i. We may assume $\|q_j\| \neq 0$. Define a bounded sequence of functions in E_i by putting $w_j = q_j/\|q_j\|$. Therefore there exist a subsequence $\{w_{j_k}\}$ and $w \in E_i$, such that $w_{j_k} \to w$ uniformly on $[0,1]$ where $\|w\| = 1$. Let $U = \{t \in [0,1] : |w(t)| > 2\varepsilon\}$. Since $\{t \in [0,1] : w(t) = 0\}$ is a finite set and w is continuous we can choose $\varepsilon > 0$ sufficiently small so that $\operatorname{meas} U > \gamma$. Then

$$\begin{aligned} |q_{j_k}(t)| &= \|q_{j_k}\||w_{j_k}(t)| \geq \|q_{j_k}\|(|w(t)| - |w(t) - w_{j_k}(t)|) \\ &\geq \|q_{j_k}\|(2\varepsilon - |w(t) - w_{j_k}(t)|) \to \infty \end{aligned}$$

uniformly for $t \in U$. $\square$

We make the following hypotheses on V:

(V′1) $0 = V(0) < \mathcal{H}(V)$ (defined in (1.3));

(V′2) there exist K, M, $G \geq 0$ and $0 < \gamma < 2$ such that

$$\langle z, \partial_z V(z,y)\rangle \leq K + MV(x) + G|x|^\gamma$$

for all $x = (z,y) \in \mathbb{R}^{n+m}$.

The hypotheses (V′1) and (V′2) do not determine whether V is bounded or unbounded and (V′2) is satisfied even if V and ∇V are both bounded. Observation (a) in the Introduction is a special case of the next result.

Theorem 3.9. *Suppose V satisfies hypotheses (V′1) and (V′2). Then there exists a solution of* (1.1) *for almost all $h \in (0, \mathcal{H}(V))$.*

Proof. First we observe that (T1) holds because $S(Y) \subset Y$, $Y = Z^\perp$ and S is self-adjoint. That (T4) holds follows trivially from the definitions of S and E_i.

To see that (T2) holds, let $q = (z, 0) \in Z$. Then Wirtinger's inequality and the definition of Z imply that there exists $c_0 > 0$, independent of $q \in Z$, such that

$$\tau(q,q) = \int_0^1 |z'(t)|^2 \, \mathrm{d}t \geq c_0 \int_0^1 |z(t)|^2 + |z'(t)|^2 \, \mathrm{d}t = c_0 \|q\|^2.$$

To show (T3), note that $q \in Y$ implies that $q = (c, y)$ where c is a constant and therefore

$$\tau(q,q) = -\int_0^1 |y'(t)|^2 \, \mathrm{d}t \leq 0.$$

Next we verify hypotheses (V). For the purpose of applying Theorem 2.7, let $0 < h_1 < h_2 < \mathcal{H}(V)$ be arbitrary but fixed. The assumption $V(0) = 0$ ensures $\mathcal{V}(0) = 0$.

To show that (V1) holds, define $g : [0, \infty) \to [0, \infty)$ by

$$g(\rho) = \inf_{|x| \geq \rho} V(x).$$

Note that g is positive, increasing and by (V′1), $\lim_{\rho \to \infty} g(\rho) = \mathcal{H}(V)$. Suppose $(c, y) + \mu(e, 0) \in \Lambda(h_2)$ with $\mu > 0$ and $c \in \mathbb{R}^n$. Then by definition of $\Lambda(h_2)$

$$\begin{aligned} h_2 &> \int_0^1 V(c + \mu e, y) \, \mathrm{d}t \geq \int_0^1 g(|(c + \mu e, y)|) \, \mathrm{d}t \\ &= \int_0^1 g\Big(\big(\sum_{i=1}^n |c_i + \mu \delta_{1,i} \cos \pi t|^2 + \sum_{i=1}^m |y_i(t)|^2 \big)^{\frac{1}{2}} \Big) \, \mathrm{d}t \\ &\geq \int_0^1 g(|c_1 + \mu \cos \pi t|) \, \mathrm{d}t, \end{aligned}$$

where $c_1 \in \mathbb{R}$. Suppose that there exists sequences $\{\mu_k\}, \{c_{1,k}\} \subset \mathbb{R}$ with $\mu_k \to \infty$ as $k \to \infty$. Then $|c_{1,k} + \mu_k \cos \pi t| \to \infty$ as $k \to \infty$ and, as in Lemma 3.8, there exists a set $U \subset [0, 1]$ with $\operatorname{meas} U > h_2 / \mathcal{H}(V)$ and an increasing subsequence $\{k_j\} \subset \mathbb{N}$ such that $|c_{1,k_j} + \mu_{k_j} \cos \pi t| \to \infty$ uniformly for $t \in U$ as $j \to \infty$. Hence

$$h_2 \geq \int_0^1 g(|c_{1,k_j} + \mu_{k_j} \cos \pi t|) \, \mathrm{d}t \geq \int_U g(|c_{1,k_j} + \mu_{k_j} \cos \pi t|) \, \mathrm{d}t \to \mathcal{H}(V) \operatorname{meas} U$$

as $j \to \infty$, which contradicts the fact that $\operatorname{meas} U > h_2 / \mathcal{H}(V)$. This shows that (V1) is satisfied by $\mathcal{V}$.

To see that (V2) holds suppose that there exists $\{q_k\} \subset \Lambda(h_2) \cap E_i$, for fixed $i \in \mathbb{N}_0$, such that $\|q_k\| \to \infty$ as $k \to \infty$. By Lemmas 3.7 and 3.8 there exists an increasing sequence $\{k_j\} \subset \mathbb{N}$ and $U \subset [0,1]$ with $\operatorname{meas} U > h_2/\mathcal{H}(V)$ such that $|q_{k_j}(t)| \to \infty$ uniformly for $t \in U$. Then

$$h_2 > \int_0^1 V(q_{k_j}(t))\,\mathrm{d}t \geq \int_U g(|q_{k_j}(t)|)\,\mathrm{d}t \to \mathcal{H}(V)\operatorname{meas} U$$

as $j \to \infty$, which contradicts the fact that $\operatorname{meas} U > h_2/\mathcal{H}(V)$. This completes the proof that hypotheses (T) and (V) are satisfied and ensures that Theorem 2.1 applies to the brake-periodic-orbit problem. Now we verify hypotheses (H1)-(H3).
(H1) Let $\gamma_2 > \gamma_1 > 0$, $D > 0$ be arbitrary and fixed. Consider the set

$$R = \cup_{i\in\mathbb{N}}\{q \in \Lambda_i(h) : \nabla_i\mathcal{J}(h,q) = 0, \gamma_1 \leq \mathcal{J}(h,q) \leq \gamma_2, 0 < \tau(q,q) \leq D\}. \tag{3.3}$$

Let $q \in R$, then

$$\tau(q,q) = \int_0^1 \langle Sq'(t), q'(t)\rangle\,\mathrm{d}t \leq D \tag{3.4}$$

and

$$\gamma_1 \leq \mathcal{J}(h,q) = \int_0^1 \langle Sq'(t), q'(t)\rangle\,\mathrm{d}t \int_0^1 h - V(q(t))\,\mathrm{d}t \leq \gamma_2. \tag{3.5}$$

Now $\langle\nabla\mathcal{J}(h,q), q\rangle = 0$ and $\mathcal{J}(h,q) > \gamma_1 > 0$ with $q = (z,y)$ together imply that

$$\begin{aligned} 2\int_0^1 h - V(q(t))\,\mathrm{d}t &= \int_0^1 \langle\nabla V(q(t)), q(t)\rangle\,\mathrm{d}t \\ &= \int_0^1 \langle z(t), \partial_z V(q(t))\rangle + \langle y(t), \partial_y V(q(t))\rangle\,\mathrm{d}t. \end{aligned} \tag{3.6}$$

Since $S(E_i) \subset E_i$ for all $i \in \mathbb{N}_0$, $\langle\nabla\mathcal{J}(h,q), Sq\rangle = 0$ and so

$$\begin{aligned} \int_0^1 h - V(q(t))\,\mathrm{d}t \int_0^1 |q'(t)|^2\,\mathrm{d}t \int_0^1 \langle Sq'(t), q'(t)\rangle\,\mathrm{d}t \int_0^1 \langle\nabla V(q(t)), Sq(t)\rangle\,\mathrm{d}t \\ = \frac{1}{2}\int_0^1 \langle Sq'(t), q'(t)\rangle\,\mathrm{d}t \int_0^1 \langle\nabla V(q(t)), Sq(t)\rangle\,\mathrm{d}t. \end{aligned}$$

Therefore, since $\mathcal{J}(h,q) > 0$,

$$\int_0^1 |q'(t)|^2\,\mathrm{d}t = \frac{\int_0^1 \langle Sq'(t), q'(t)\rangle\,\mathrm{d}t \int_0^1 \langle \nabla V(q(t)), Sq(t)\rangle\,\mathrm{d}t}{2\int_0^1 h - V(q(t))\,\mathrm{d}t}$$
$$= \frac{\left(\int_0^1 \langle Sq'(t), q'(t)\rangle\,\mathrm{d}t\right)^2 \int_0^1 \langle \nabla V(q(t)), Sq(t)\rangle\,\mathrm{d}t}{2\mathcal{J}(h,q)} \tag{3.7}$$

and, from (3.4), (3.5), (3.6) and since $q \in \Lambda(h)$,

$$\begin{aligned}\int_0^1 |q'(t)|^2\,\mathrm{d}t &\leq \frac{D^2}{2\gamma_1}\int_0^1 \langle \nabla V(q(t)), Sq(t)\rangle\,\mathrm{d}t \\ &= \frac{D^2}{2\gamma_1}\int_0^1 \langle z(t), \partial_z V(q(t))\rangle - \langle y(t), \partial_y V(q(t))\rangle\,\mathrm{d}t \\ &= \frac{D^2}{\gamma_1}\int_0^1 \langle z(t), \partial_z V(q(t))\rangle - h + V(q(t))\,\mathrm{d}t \\ &< \frac{D^2}{\gamma_1}\int_0^1 \langle z(t), \partial_z V(q(t))\rangle\,\mathrm{d}t.\end{aligned}$$

This together with hypothesis (V′2) yields

$$\begin{aligned}\int_0^1 |q'(t)|^2\,\mathrm{d}t &< \frac{D^2}{\gamma_1}\left(K + Mh + G\int_0^1 |q(t)|^\gamma\,\mathrm{d}t\right) \\ &< \frac{D^2}{\gamma_1}\left(K + Mh + G\left(\int_0^1 |q(t)|^2\,\mathrm{d}t\right)^{\frac{\gamma}{2}}\right),\end{aligned} \tag{3.8}$$

by Jensen's inequality. Since $q \in \Lambda(h)$,

$$\int_0^1 g(|q(t)|)\,\mathrm{d}t \leq \int_0^1 V(q(t))\,\mathrm{d}t < h$$

where $\lim_{\rho\to\infty} g(\rho) = \mathcal{H}(V) > h$ as $r \to \infty$. Therefore

$$\sup_{q\in\Lambda(h)}\ \min_{t\in[0,1]} |q(t)| = P < \infty$$

and so

$$|q(t)|^2 \leq P^2 + 2\|q\|_{L^2}\|q'\|_{L^2}. \tag{3.9}$$

Combining (3.9) and (3.8) yields that R is bounded in X. Therefore (H1) holds.

From the definition of $N : X \to X$,

$$\langle N(q), v\rangle = \int_0^1 \langle Sq'(t), q'(t)\rangle \,\mathrm{d}t \int_0^1 \langle \nabla V(q(t)), v(t)\rangle \,\mathrm{d}t,$$

for all $q, v \in X$. Let $\{q_i\}$ be bounded in X. Choose a subsequence, relabelled $\{q_i\}$, such that

$$\begin{aligned}
&q_i \rightharpoonup q \text{ for some } q \in X,\\
&\int_0^1 \langle Sq_i'(t), q_i'(t)\rangle \,\mathrm{d}t \to s \text{ for some } s \in \mathbb{R},\\
&q_i \to q,\ \nabla V(q_i) \to \nabla V(q) \text{ in } L^\infty.
\end{aligned}$$

Then

$$\begin{aligned}
&|\langle N(q_i) - N(q_k), v\rangle|\\
&\leq \left|\int_0^1 \langle Sq_i'(t), q_i'(t)\rangle - \langle Sq_k'(t), q_k'(t)\rangle \,\mathrm{d}t\right| \left|\int_0^1 \langle \nabla V(q_k(t)), v(t)\rangle \,\mathrm{d}t\right|\\
&\qquad + \left|\int_0^1 \langle Sq_k'(t), q_k'(t)\rangle \,\mathrm{d}t\right| \left|\int_0^1 \langle \nabla V(q_i(t)) - \nabla V(q_k(t)), v(t)\rangle \,\mathrm{d}t\right|\\
&= \varepsilon_{i,k}\|v\| \text{ where } \varepsilon_{i,k} \to 0 \quad \text{as } i,\ k \to \infty.
\end{aligned}$$

Therefore $\|N(q_i) - N(q_k)\| \to 0$ as i, $k \to \infty$ and (H2) holds.

Finally to show (H3), suppose that $\nabla\mathcal{J}(h, q_i) \to 0$ in X^* as $i \to \infty$, where $\{q_i\} \subset R$. Recall that for $v \in X$,

$$\begin{aligned}
\langle \nabla\mathcal{J}(h, q_i), v\rangle &= 2\int_0^1 \langle Sq_i'(t), v'(t)\rangle \,\mathrm{d}t \int_0^1 h - V(q_i(t)) \,\mathrm{d}t\\
&\quad - \int_0^1 \langle Sq_i'(t), q_i'(t)\rangle \,\mathrm{d}t \int_0^1 \langle \nabla V(q_i(t)), v(t)\rangle \,\mathrm{d}t.
\end{aligned}$$

Since $\{q_i\} \subset R$, which is bounded in X, we may suppose, without loss of generality, that (for a subsequence)

$$q_i \rightharpoonup q \text{ in } X, \quad q_i \to q \text{ in } L^\infty, \quad \nabla V(q_i) \to \nabla V(q) \text{ in } L^\infty. \tag{3.10}$$

Since $S(q_i - q)$ is bounded in X (because R is bounded) and $\nabla\mathcal{J}(h, q_i) \to 0$ in

X^* it follows that

$$2\int_0^1 \langle Sq_i'(t), S(q_i'(t)-q'(t))\rangle \,\mathrm{d}t \int_0^1 h - V(q_i(t))\,\mathrm{d}t \\ - \int_0^1 \langle Sq_i'(t), q_i'(t)\rangle \,\mathrm{d}t \int_0^1 \langle \nabla V(q_i(t)), S(q_i(t)-q(t))\rangle \,\mathrm{d}t \to 0$$

as $i \to \infty$. Therefore, for a subsequence,

$$K_1 \int_0^1 \langle Sq_i'(t), S(q_i'(t)-q'(t))\rangle \,\mathrm{d}t - K_2 \int_0^1 \langle \nabla V(q_i(t)), S(q_i(t)-q(t))\rangle \,\mathrm{d}t \to 0$$

as $j \to \infty$, for some $K_1 \geq 2\gamma_1/D$ and $0 < K_2 \leq D$. It follows from this observation and (3.10) that

$$\int_0^1 \langle q_i'(t), q_i'(t)-q'(t)\rangle \,\mathrm{d}t \to 0,$$

and, since $q_i' \rightharpoonup q'$ in L^2,

$$\int_0^1 |q_i'(t)|^2 \,\mathrm{d}t \to \int_0^1 |q'(t)|^2 \,\mathrm{d}t.$$

But $q_i' \rightharpoonup q'$ in L^2 and $\|q_i'\|_{L^2} \to \|q'\|_{L^2}$ together imply that $q_i' \to q'$ in L^2. Now

$$\|q-q_j\|_{L^2}^2 \leq \|q-q_j\|_{L^\infty} + 2\|q-q_j\|_{L^2}\|q'-q_j'\|_{L^2}$$

together with $q_i \to q$ in L^∞ implies $q_i \to q$ in L^2. Hence $q_i \to q$ in X, as required. Thus (H3) holds.

We have shown that all the conditions of Theorem 2.7 are satisfied. Therefore there exists $\beta_1, \beta_2 \in \mathbb{R}$ such that for almost all $h \in [h_1, h_2]$ there is a $q \in X$ satisfying

$$q \in \Lambda(h), \quad \nabla\mathcal{J}(h,q) = 0, \; 0 < \beta_1 \leq \mathcal{J}(h,q) \leq \beta_2 \;\text{ and }\; \tau(q,q) \leq \alpha, \tag{3.11}$$

where β_1 and β_2 are independent of $h \in [h_1, h_2]$ and α, given by (2.7), is dependent on h. Then by Lemma 3.1 there exists a solution of (1.1) for almost all $h \in [h_1, h_2]$. Since h_1 and h_2 were chosen arbitrarily in $(0, \mathcal{H}(V))$ the proof is complete. $\square$

Corollary 3.10. *If V satisfies (V′2) and $0 \leq V(x) < \mathcal{H}(V)$ for all $x \in X$, there* *exists a brake periodic orbit for almost all $h \in (0, \mathcal{H}(V))$ and no brake periodic orbit for any $h \geq \mathcal{H}(V)$.*

Proof. By Theorem 2.7 there exists a brake periodic orbit for almost all $h \in (0, \mathcal{H}(V))$. By (1.1b) and (1.1c), the energy of a brake periodic orbit must lie in the range of V. Hence, by the hypothesis on V, no such orbits exist with $h \geq \mathcal{H}(V)$. □

We now introduce another hypothesis on V which can replace (V′2).

(B1) there exists $\mathcal{N} \subset \mathbb{R}^{n+m}$ and $B \geq 1$ such that $\mathbb{R}^{n+m} \setminus \mathcal{N}$ is compact and

$$\langle \nabla V(x), Sx \rangle \leq B \langle \nabla V(x), x \rangle \quad \text{for all } x \in \mathcal{N}.$$

Condition (B1) allows V to have arbitrary growth which is essential when considering singular potentials in Chapter 5.

Theorem 3.11. *Theorem 3.9 remains valid when hypothesis (V′2) is replaced by hypothesis (B1).*

Proof. In the proof of Theorem 3.9, hypothesis (V′2) is only used in the verification of (H1). It is sufficient to prove that $\sup_{q \in R} \|q'\|_{L^2} < \infty$ where R is defined on page 36. Let $q \in R$ then by (3.7)

$$\begin{aligned}\int_0^1 |q'(t)|^2 \, \mathrm{d}t &= \frac{\left(\int_0^1 \langle Sq'(t), q'(t) \rangle \, \mathrm{d}t\right)^2 \int_0^1 \langle \nabla V(q(t)), Sq(t) \rangle \, \mathrm{d}t}{2\mathcal{J}(h,q)} \\ &\leq \frac{D^2}{2\gamma_1} \int_0^1 \langle \nabla V(q(t)), Sq(t) \rangle \, \mathrm{d}t \quad \text{by (3.4) and (3.5)}\end{aligned}$$

Since $\mathbb{R}^{n+m} \setminus \mathcal{N}$ is compact,

$$C := \max\{\langle \nabla V(x), Sx \rangle, -\langle \nabla V(x), x \rangle : x \in \mathbb{R}^{n+m} \setminus \mathcal{N}\} < \infty. \tag{3.12}$$

Therefore

$$
\begin{aligned}
\int_0^1 |q'(t)|^2 \,\mathrm{d}t &\le \frac{D^2}{2\gamma_1} \int_0^1 \langle \nabla V(q(t)), Sq(t)\rangle \,\mathrm{d}t \\
&= \frac{D^2}{2\gamma_1} \left\{ \int_{\{t: q(t) \in \mathbb{R}^{n+m} \setminus \mathcal{N}\}} \langle \nabla V(q(t)), Sq(t)\rangle \,\mathrm{d}t + \int_{\{t: q(t)\in \mathcal{N}\}} \langle \nabla V(q(t)), Sq(t)\rangle \,\mathrm{d}t \right\} \\
&\le \frac{D^2}{2\gamma_1} \left\{ C + B \int_{\{t: q(t)\in\mathcal{N}\}} \langle \nabla V(q(t)), q(t)\rangle \,\mathrm{d}t \right\} \quad \text{by (3.12) and (B1),} \\
&= \frac{D^2}{2\gamma_1} \left\{ C + B \left(\int_0^1 \langle \nabla V(q(t)), q(t)\rangle \,\mathrm{d}t - \int_{\{t: q(t) \in \mathbb{R}^{n+m}\setminus\mathcal{N}\}} \langle \nabla V(q(t)), q(t)\rangle \,\mathrm{d}t \right) \right\} \\
&\le \frac{D^2}{2\gamma_1} \left\{ C + B\,(2h + C) \right\} \quad \text{by (3.6) and (3.12).}
\end{aligned}
$$

Hence $\sup_{q\in R} \|q'\|_{L^2} < \infty$ as required. □

3.3 The period is related to the derivative of the minimax function

Existence of brake periodic orbits of (1.1) for almost all $h \in (0, \mathcal{H}(V))$ have been established in Theorem 3.9. In particular, for each $h \in (0, \mathcal{H}(V))$ for which existence is a consequence of Theorem 3.9, (3.11) gives

$$q \in \Lambda(h), \quad \nabla\mathcal{J}(h,q) = 0, \quad 0 < \beta_1 \le \mathcal{J}(h,q) \le \beta_2 \quad \text{and} \quad \tau(q,q) \le \alpha,$$

where $\alpha = \liminf_{i\to\infty} c_i'(h, \underline{h})$ as defined in (2.7). By Lemma 3.1, q can be rescaled in time to give a solution of (1.1) with $t_1 = 0$ and

$$
\begin{aligned}
t_0^2 &= \frac{\mathcal{J}(h,q)}{2\,(h - \mathcal{V}(q))^2} = \frac{\tau(q,q)^2}{2\mathcal{J}(h,q)} \quad \text{by (3.1)} \\
&\le \frac{1}{2\beta_1} \liminf_{i\to\infty} c_i'(h, \underline{h})^2.
\end{aligned}
$$

The value t_0 is a multiple of the half-period of the orbit. A stronger result relating period to energy was proved by van Groesen [vanG86] in the case when S is positive definite and under assumptions that included the convexity of V.

3.4 Non-existence for energies in a set of zero measure

Here we seek to say that there are no solutions with energies from a wide class of prescribed sets of zero energy. Recall that the set $\mathcal{A}(V)$ defined by (1.4) is relatively closed with zero measure. The following discussion is based on [BatNor96]. The complement of a compact set C in $\mathbb{R}$ is a countable union of disjoint open intervals, the bounded intervals are called the gaps of C and $\mathcal{G}_C$ denotes the set of gaps. For $\alpha > 0$ the degree-α gap sum of C is

$$G_\alpha(C) = \sum_{I \in \mathcal{G}_C} |I|^\alpha, \quad \text{where} \quad |I| = meas\, I.$$

Theorem 3.12. *Let $0 < H \leq \infty$ and let $A \subset [0, H)$ be closed relative to $[0, H)$ with $0 \in A$, $meas\, A = 0$ and $G_{1/2}(\overline{A} \cap I) < \infty$, whenever I is a compact interval in $\mathbb{R}$. Then there exists a C^2-function $V : \mathbb{R}^{m+n} \to [0, H)$, with $V(0) = 0$, $V(x) \to H$ as $|x| \to \infty$ and ∇V bounded, such that there are no solutions of* (1.1) *with $h \in A$. (In particular, (V'1) and (V'2) hold, $\mathcal{H}(V) = H$ and $\mathcal{A}(V) = A$.)*

Proof. Let $V(x) = f(|x|)$, $x \in \mathbb{R}^{n+m}$ where f is the function whose existence is established in Lemma 3.15. Now suppose that there exists a solution of (1.1) with $h \in A$. Then (1.1c) and (1.1b) imply that $V(q(0)) = h \in A$ and hence $\nabla V(q(0)) = 0$. It follows that the constant $q(0)$ satisfies (1.1a) and (1.1b). Since V is C^2 the uniqueness theorem for initial value problems gives $q(t) \equiv q(0)$ for all t. Therefore, for this choice of potential energy function V, (1.1) has no brake periodic solutions with energy $h \in A$. □

Lemma 3.13. *Let $B \subset \mathbb{R}$ be compact and $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and n times continuously differentiable on $\mathbb{R} \setminus B$. Suppose*

(a) $|f^{(i)}(x)| \to 0$ *uniformly as $x \to B$, $x \in \mathbb{R} \setminus B$, $i = 1, \dots, n$,*

(b) $|f(x) - f(y)| = o(|x - y|)$ *as $|x - y| \to 0$, uniformly for all $x, y \in B$,*

where $f^{(i)}$ denotes the i^{th} derivative of f. Then $f \in C^n(\mathbb{R})$.

Proof. Let $\varepsilon > 0$. By (a) and (b) there exists $\delta > 0$ (independent of i) such that for all $i \in \{1, \dots, n\}$; $|f^{(i)}(x)| < \varepsilon$ whenever $x \in \mathbb{R} \setminus B$ and $\text{dist}(x, B) < \delta$; and

$|f(x)-f(y)| < \varepsilon|x-y|$ whenever $x, y \in B$ and $|x-y| < \delta$. Fix $x \in B$ and let $y \in \mathbb{R}$ be such that $0 < |x-y| < \delta$. Suppose $y > x$ and let $b = \max\{z \in B : z \leq y\}$. (If $y < x$ then consider $b = \min\{z \in B : z \geq y\}$.) Then since B is compact and $x \in B$, $b \in B$ and $b \leq y$. Note that if $b \neq y$, $(b, y) \subset \mathbb{R} \setminus B$, so f is continuous on $[b, y]$ and n times differentiable on (b, y). Moreover, by (a), each of the n derivatives may be extended to continuous functions on $[b, y]$. Therefore the fundamental theorem of calculus applies to $f^{(i)}|_{[b,y]}$ for each $i \in \{1, \ldots, n\}$.

We now claim that $f^{(i)}(x) = 0$ for all $x \in B$ and $i \in \{1, \ldots, n\}$. To prove the case $i = 1$ note that

$$\begin{aligned}
|f(y) - f(x)| &\leq |f(y) - f(b)| + |f(b) - f(x)| \\
&= \left| \int_b^y f'(t)\,\mathrm{d}t \right| + |f(b) - f(x)| \\
&\leq \varepsilon(y-b) + \varepsilon(b-x) = \varepsilon(y-x)
\end{aligned}$$

and therefore $f'(x) = 0$ for all $x \in B$. Now suppose that the claim holds for some $i \in \{1, \ldots, n-1\}$. That is $f^{(i)}(x) = 0$ for all $x \in B$. Then

$$\begin{aligned}
|f^{(i)}(y) - f^{(i)}(x)| &\leq |f^{(i)}(y) - f^{(i)}(b)| + |f^{(i)}(b) - f^{(i)}(x)| \\
&= |f^{(i)}(y) - f^{(i)}(b)| \\
&= \left| \int_b^y f^{(i+1)}(t)\,\mathrm{d}t \right| \leq \varepsilon|y-b| \leq \varepsilon|y-x|.
\end{aligned}$$

Hence $f^{(i+1)}(x) = 0$ for all $x \in B$. Therefore $f^{(n)}(x) = 0$ for all $x \in B$ by induction. This combined with (a) yields that $f \in C^n(\mathbb{R})$. ☐

Lemma 3.14. *Let $A \subset [0, \infty)$ be compact with $\operatorname{meas} A = 0$, $0 \in A$ and $\mathcal{G}_{1/2}(A) < \infty$. Then there exists $f : [0,1] \to [0, \sup A]$ such that f is C^2, onto, strictly increasing, $f'(t) = 0$ if and only if $f(t) \in A$, and $f''(t) = 0$ if $f(t) \in A$.*

Proof. Let $\mathcal{G}_A = \{I_i : i \in \mathbb{N}\}$ be the gaps of A and put $\mathcal{G}_x = \{i \in \mathbb{N} : \sup I_i \leq x\}$. The hypothesis $G_{1/2}(A) < \infty$ can be written $\sum_{i \in \mathbb{N}} |I_i|^{1/2} < \infty$.

The construction of the following sequence $\{r_k\}$ which follows is due to du Bois Reymond [Bro42, Art. 16(5), page 47]. For $k \in \mathbb{N}$, put $r_k^{-1} = \sum_{i=k}^{\infty} |I_i|^{1/2}$.

Then $\{r_k\}$ is an unbounded monotone sequence of positive terms with

$$0 \leq \sqrt{r_k}\,|I_k|^{1/2} = \frac{r_{k+1} - r_k}{r_{k+1}\sqrt{r_k}} \leq 2\left(\frac{1}{\sqrt{r_k}} - \frac{1}{\sqrt{r_{k+1}}}\right).$$

(The inequality follows from the convexity observation that $2(1-\sqrt{x}) \geq 1-x$, $x \in (0,1)$, by putting $x = r_k/r_{k+1}$.) Since the right hand side is the k^{th} positive term of a series with sum $2/\sqrt{r_1}$, the left side (by the comparison test) is the k^{th} term of a series with finite sum, r, say. Let $m_i = \sqrt{r_i}/r$. Then $m_i \to \infty$ as $i \to \infty$ and $\sum_{i \in \mathbb{N}} m_i |I_i|^{1/2} = 1$.

Now define $h : A \to \mathbb{R}$ by $h(x) = \sum_{i \in \mathcal{G}_x} m_i |I_i|^{1/2}$. For $x, y \in A$ with $x < y$ there holds $\mathcal{G}_x \subset \mathcal{G}_y$ without equality. So h is strictly increasing on A. Since $\mathcal{G}_{\sup A} = \mathbb{N}$ and $0 \in A$ there holds respectively, $h(\sup A) = 1$ and $h(0) = 0$.

Suppose $x \in A$ is not an isolated point and let $x_n \in A$ be such that $x_n \to x$ as $n \to \infty$. Define $H, H_n, G : \mathbb{N} \to [0, \infty)$ pointwise by

$$H(i) = \mathcal{X}_{\mathcal{G}_x}(i)\, m_i |I_i|^{1/2}, \quad H_n(i) = \mathcal{X}_{\mathcal{G}_{x_n}}(i)\, m_i |I_i|^{1/2}, \quad G(i) = m_i |I_i|^{1/2}.$$

Note that for each $i \in \mathbb{N}$, $H_n(i) \to H(i)$ as $i \to \infty$ and, moreover, for each $n \in \mathbb{N}$, H_n is dominated by G. Since $G \in L^1(\mathbb{N}, \mathrm{d}i)$, the Dominated Convergence Theorem applies to yield

$$h(x_n) = \sum_{i \in \mathcal{G}_{x_n}} m_i |I_i|^{1/2} = \int_{\mathbb{N}} H_n(i)\,\mathrm{d}i \to \int_{\mathbb{N}} H(i)\,\mathrm{d}i = \sum_{i \in \mathcal{G}_x} m_i |I_i|^{1/2} = h(x),$$

giving that h is continuous. If we put $B = h(A)$ then B is a compact subset of $[0,1]$ with $\operatorname{meas} B = 0$. Also, $h : A \to B$ is a continuous bijection between compact sets, implying that h is a homeomorphism. Denote the inverse of h by g.

Let $x, y \in A$ with $x \neq y$, and let $\mathcal{D}_{x,y} = (\mathcal{G}_x \setminus \mathcal{G}_y) \cup (\mathcal{G}_y \setminus \mathcal{G}_x) \subset \mathbb{N}$. Then $\min \mathcal{D}_{x,y} \to \infty$ uniformly as $|x - y| \to 0$, otherwise there exists $N \in \mathbb{N}$ and sequences $\{x_n\}$, $\{y_n\}$ such that $|x_n - y_n| \to 0$ as $n \to \infty$ with $\min \mathcal{D}_{x_n, y_n} \leq N$ for all $i \in \mathbb{N}$. Then one of $\{I_1, \ldots, I_N\}$ is contained in (x_n, y_n) for all $n \in \mathbb{N}$, which is a contradiction.

Then for $x, y \in A$ with $|x-y| < 1$, there holds $|I_i| < 1$ for all $i \in \mathcal{D}_{x,y}$ and

$$\frac{|h(x)-h(y)|}{|x-y|} = \frac{\sum_{i \in \mathcal{D}_{x,y}} m_i |I_i|^{1/2}}{\sum_{i \in \mathcal{D}_{x,y}} |I_i|} \geq \frac{\sum_{i \in \mathcal{D}_{x,y}} m_i |I_i|}{\sum_{i \in \mathcal{D}_{x,y}} |I_i|}$$
$$\geq \min\{m_i : i \in \mathcal{D}_{x,y}\} = m_{\min \mathcal{D}_{x,y}}.$$

Therefore, $|g(x) - g(y)| = o(|x-y|)$ uniformly for all $x, y \in B$ as $|x-y| \to 0$.

Since h is bijective, there is a one-to-one correspondence between gaps of A and gaps of B. If $I_i \in \mathcal{G}_A$, denote the corresponding gap of B by (a_i, b_i). By the definition of g, $b_i - a_i = m_i |I_i|^{1/2}$. Since $\operatorname{meas} B = 0$ and $B \subset [0,1]$, $\sum_{i \in \mathbb{N}} (b_i - a_i) = 1$ and $B = \bigcup_{i \in \mathbb{N}} \{a_i, b_i\}$.

Let $s : [0,1] \to [0, \infty)$ be a smooth function such that; $s(t) > 0$ whenever $t \in (0,1)$, $s^{(i)}(0) = 0 = s^{(i)}(1)$ for all $i \in \mathbb{N} \cup \{0\}$ and $\int_0^1 s(t)\,\mathrm{d}t = 1$.

Define $F : \mathbb{R} \to [0, \infty)$ by

$$F(t) = \begin{cases} 0, & t \in \mathbb{R} \setminus \bigcup_{i \in \mathbb{N}} (a_i, b_i), \\ \frac{|I_i|^{1/2}}{m_i} s\left(\frac{t - a_i}{m_i |I_i|^{1/2}}\right), & t \in \bigcup_{i \in \mathbb{N}} (a_i, b_i), \end{cases}$$

and let $f : \mathbb{R} \to [0, \infty)$ be defined by $f(x) = \int_0^x F(t)\,\mathrm{d}t$. Note that f is strictly increasing on $[0,1]$, smooth on $\mathbb{R} \setminus B$, continuous on $\mathbb{R}$ and $f(x) = 0$ for $x \leq 0$.

Suppose $x \in B = \bigcup_{i \in \mathbb{N}} \{a_i, b_i\}$ then

$$f(x) = \int_0^x F(t)\,\mathrm{d}t = \sum_{i \in \mathcal{G}_{g(x)}} \frac{|I_i|^{1/2}}{m_i} \int_{a_i}^{b_i} s\left(\frac{t - a_i}{m_i |I_i|^{1/2}}\right) \mathrm{d}t$$
$$= \sum_{i \in \mathcal{G}_{g(x)}} \frac{|I_i|^{1/2}}{m_i} m_i |I_i|^{1/2} = \sum_{i \in \mathcal{G}_{g(x)}} |I_i| = g(x).$$

Therefore $|f(x) - f(y)| = o(|x-y|)$ uniformly for $x, y \in B$ as $|x-y| \to 0$.

By the definition of $s : [0,1] \to [0, \infty)$ there exists $\alpha > 0$ such that, for all $t \in [0,1]$,

$$\max\{s(t), |s'(t)|\} \leq \alpha \max\{t, 1-t\}.$$

Fix $\varepsilon > 0$ then for $t \in (a_i, b_i)$,

$$\begin{aligned}
F(t) &= \frac{|I_i|^{1/2}}{m_i} s\left(\frac{t-a_i}{m_i|I_i|^{1/2}}\right) \leq \frac{\alpha}{m_i}|I_i|^{1/2}\max\{\frac{t-a_i}{m_i|I_i|^{1/2}}, 1-\frac{t-a_i}{m_i|I_i|^{1/2}}\} \\
&= \frac{\alpha}{m_i^2}\max\{t-a_i, m_i|I_i|^{1/2}-(t-a_i)\} \\
&= \frac{\alpha}{m_i^2}\operatorname{dist}(t, \{a_i, b_i\}) < \varepsilon,
\end{aligned}$$

whenever $\operatorname{dist}(t, B) < \varepsilon\alpha^{-1}m_1^2$. Hence $|f'(t)| = F(t) \to 0$ as $\operatorname{dist}(t, B) \to 0$.

For $\delta > 0$ let $N_\delta = \{i \in \mathbb{N} : b_i - a_i > \delta\}$. Note that N_δ is a finite subset of $\mathbb{N}$. Choose δ sufficiently small so that $\alpha\, m_{\min \mathbb{N}\setminus N_\delta}^{-2} < \varepsilon$. Now if

$$\operatorname{dist}(t, B) < \min\{\delta, \frac{\varepsilon}{\alpha}\min(m_i^3|I_i|^{1/2} : i \in N_\delta)\},$$

and $t \in (a_i, b_i)$,

$$|f^{(2)}(t)| = |F'(t)| = \frac{1}{m_i^2}\left|s'\left(\frac{t-a_i}{m_i|I_i|^{1/2}}\right)\right| \leq \frac{\alpha}{m_i^3|I_i|^{1/2}}\operatorname{dist}(t, \{a_i, b_i\}) < \varepsilon.$$

Hence $f^{(i)}(t) \to 0$ uniformly as $t \to B$ for $i \in \{1, 2\}$. Finally a direct application of Theorem 3.13 completes the proof. $\square$

Lemma 3.15. *Let A denote the set in Theorem 3.12. Then there exists a C^2-function $f : [0, \infty) \to [0, H)$ which is non-decreasing on $[0, \infty)$, with $f(0) = f'(0) = 0$, f' bounded and $f'(t) = 0$ if and only if $f(t) \in A$.*

Proof. Let $K = \sup A$ and consider first the case when $K < \infty$. Then $\{K\} \cup A = \overline{A}$ is compact. Note that K does not necessarily lie in A. By Lemma 3.14 there exists a C^2-function $u : [0, 1] \to [0, K]$ such that u is non-decreasing, onto and $u'(t) = 0$ if and only if $u(t) \in \{K\} \cup A$. Moreover $u''(t) = 0$ if $u(t) \in \{K\} \cup A$. If $K \notin A$ then $H = K < \infty$ and $f = u \circ g$, where $g : [0, \infty) \to [0, 1)$ is smooth, onto, $g' > 0$ and $g(0) = 0$. If $K \in A$ then $K < H < \infty$ and u can be extended as a C^2 function from $[0, \infty)$ onto $[0, H)$, for which f' has no zeros on $[0, \infty) \setminus f^{-1}(A)$, and the proof is complete.

It remains to consider the case when $K = \infty$. Let $\{K_j\}_{j \in \mathbb{N}\cup\{0\}}$ denote an increasing sequence in A with $K_0 = 0$ and $K_j \to \infty$ as $j \to \infty$. Then $A \cap [K_{j-1}, K_j]$ is compact and $\mathcal{G}_{1/2}(A \cap [K_{j-1}, K_j]) < \infty$ for all $j \in \mathbb{N}$. The construction for finite K above can be used to construct a sequence of functions $u_j : [j-1, j] \to [K_{j-1}, K_j]$, $j \in \mathbb{N}$, such that $u_j'(j-1) = u_j'(j) = u_j''(j-1) = u_j''(j) = 0$ and $u_j'(t) = 0$, $t \in [K_{j-1}, K_j]$, if and only if $u_j(t) \in A \cap [K_{j-1}, K_j]$. Let $T_0 = 0$ and for each $j \in \mathbb{N}$, let $T_j \geq 1$ be such that on $[(j-1)T_j, jT_j]$ the function f_j defined by

$$f_j(t) = u_j(t/T_j), \quad [(j-1)T_j, jT_j],$$

has derivative bounded by 1. To complete the proof when $K = \infty$ let

$$f(t) = f_{k+1}\big(t + kT_{k+1} - \sum_{j=0}^{k} T_j\ \big) \text{ if } t \in \Big[\sum_{j=0}^{k} T_j, \sum_{j=0}^{k+1} T_j\Big], \quad k \in \mathbb{N} \cup \{0\}.$$

The function so defined has bounded derivative and is strictly monotone with all the required properties. □

An example of an uncountable set satisfying our hypotheses in Theorem 3.12 is provided by the middle-3/5 Cantor set. Define C by $C = \cap_{n\geq 1} C_n$ where $C_1 = [0,1]$ and C_{n+1} is defined by C_n minus the open middle $(1-2l)$th of each of the components of C_{n+1} $(l < 1/2)$. Since C is the intersection of a nested sequence of non-empty compact sets, it is non-empty and compact. Moreover, every point of C is a limit point of C. Therefore C is a perfect set and hence uncountable. Note that $0 \in C \subset [0,1]$ and by definition, if $2l^\alpha < 1$, we have

$$G_\alpha(C) = \sum_{j=0}^{\infty} 2^j (l^j - 2l^{j+1})^\alpha = \frac{(1-2l)^\alpha}{1-2l^\alpha} < \infty.$$

Hence if $l = 1/5$ then $G_1(C) = 1$ and $G_{1/2}(C) = \sqrt{3}(\sqrt{5}-2)^{-1} < \infty$. This is an example of a perfect set of measure zero satisfying our hypotheses. Observe, by taking $l = 1/3$ and $\alpha = 1/2$, that not every perfect set satisfies the finite 1/2-gap sum condition.

3.5 Bounded even solutions for all energies

We have just seen that there maybe large sets of energies for which there are no non-constant brake-periodic orbits. In the light of Section 3.1 the following observation, which addresses the question of what happens to brake periodic orbits at nearby energy levels, is straightforward. Observation (b) of the Introduction is a consequence of the following.

Lemma 3.16. *Suppose that $h^* \in (0, \mathcal{H}(V))$ is such that there is no solution of* (1.1) *with $h = h^*$. Let $\{h_n\} \subset (0, \mathcal{H}(V))$ be any sequence with $h_n \to h^*$ for which the existence of a solution u_n of* (1.1) *with energy h_n is given by Theorem 3.9. Then $t_n \to \infty$ as $n \to \infty$ where t_n is defined by the formula for t_0 in Lemma 3.1.*

If,

$$\liminf_{|x|\to\infty} \{2(V(x) - h^*) - \langle Sx, \nabla V(x)\rangle\} > 0,$$

then a subsequence of $\{u_n\}$ converges uniformly on $[-T, T]$, for each $T > 0$, to a bounded even solution u^* of (1.1a) *and* (1.1b) *with $h = h^*$.*

Proof. Let $u_n(t) = q_n(t/t_n)$ where t_n is given by the formula for t_0 in Lemma 3.1

$$t_n^2 = \frac{\mathcal{J}(h_n, q_n)}{2(h_n - \int_0^1 V(q_n(t))\,\mathrm{d}t)^2}.$$

Suppose for a contradiction that $t_n \le M < \infty$ for all $n \in \mathbb{N}$. Then equation (3.11) in the proof of Theorem 3.9 shows that there exists β_1, $\beta_2 \in \mathbb{R}$, independent of n, such that

$$q_n \in \Lambda(h_n), \quad \nabla\mathcal{J}(h_n, q_n) = 0 \quad \text{and} \quad 0 < \beta_1 \le \mathcal{J}(h_n, q_n) \le \beta_2.$$

This gives that $0 < \tau(q_n, q_n) \le \sqrt{2M\beta_2}$. The proof of Theorem 3.9 then shows that $\{q_n\}$ has a strongly convergent subsequence with limit $q \in X$ satisfying

$$q \in \Lambda(h^*), \quad \nabla\mathcal{J}(h^*, q) = 0 \quad \text{and} \quad 0 < \beta_1 \le \mathcal{J}(h^*, q) \le \beta_2,$$

which, by Lemma 3.1, yields a non-constant brake periodic orbit with energy h^*. This is a contradiction.

By hypothesis

$$\liminf_{|x|\to\infty} \{2(V(x) - h^*) - \langle Sx, \nabla V(x)\rangle\} > 0,$$

there exists $R > 0$ and $N \in \mathbb{N}$ such that

$$\inf_{|x|>R} \{2V(x) - \langle Sx, \nabla V(x)\rangle\} > 2h_n,$$

for all $n > N$. Hence by Theorem 3.2

$$\begin{aligned} |u_n(t)| &\le \max\{R, |u_n(0)|, |u_n(t_n)|\} \\ &\le \max\{R, \sup(|x| : V(x) = h_n)\} < \infty. \end{aligned}$$

Since $\lim_{n\to\infty} h_n = h^* < \mathcal{H}(V)$, $\{u_n\}$ is bounded in L^∞. Let $T > 0$ then by (1.1a) and (1.1c)

$$\begin{aligned} \frac{1}{2}\int_0^T |u_n'(t)|^2 \,\mathrm{d}t &= \int_0^T \int_0^s \langle u'(t), u''(t)\rangle \,\mathrm{d}t\,\mathrm{d}s \\ &\le T \int_0^T |u_n'(t)||\nabla V(u_n(t))| \,\mathrm{d}t \\ &\le T \left(\int_0^T |u_n'(t)|^2 \,\mathrm{d}t\right)^{1/2} \left(\int_0^T |\nabla V(u_n(t))|^2 \,\mathrm{d}t\right)^{1/2}. \end{aligned}$$

Since ∇V is continuous and $\{u_n\}$ is bounded in L^∞(0,T), $\{u_n'\}$ is bounded in $L^2(0,T)$ and hence it is bounded in $W^{1,2}(0,T)$. Then for a subsequence, $u_n \to u$ in $L^\infty(0,T)$ and $u_n \rightharpoonup u$ in $W^{1,2}(0,T)$. Hence $\{u_n\}$ converges uniformly to a bounded solution of (1.1a) and (1.1b) with energy h^*. □

3.6 Brake periodic orbits for all positive energies

To obtain existence for all $h > 0$ it is clear from Theorem 3.12 that we need an extra assumption on V. The following sufficient condition contrasts with that of the previous section.

(V′3) There exists $p > 0$ such that

$$\langle \nabla V(x), x\rangle \ge pV(x) > 0 \text{ for all } x \in \mathbb{R}^{n+m} \setminus \{0\}.$$

Theorem 3.17. *Suppose that V satisfies (V'2) and (V'3). Then, for all $h > 0$, there exists a non-constant solution of* (1.1).

Proof. Note that (V'3) ensures $\liminf_{|z|\to\infty} V(z) = \infty$ so that (V'1) holds with $\mathcal{H}(V) = \infty$. Choose $h > 0$ and set $h_1 = h$ and $h_2 = h + 1$. Then by (3.11) in the proof of Theorem 3.9, there exists $q_n \in X$ and $h_n \in [h, h+1]$ such that $h_n \to h$ and

$$q_n \in \Lambda(h_n), \quad \nabla\mathcal{J}(h_n, q_n) = 0 \quad \text{and} \quad 0 < \beta_1 \le \mathcal{J}(h_n, q_n) \le \beta_2,$$

for all $n \in \mathbb{N}$, where $0 < \beta_1 < \beta_2$ are independent of n. Since $\langle \nabla\mathcal{J}(h_n, q_n), q_n\rangle = 0$ and $\tau(q_n, q_n) > 0$,

$$2\int_0^1 h_n - V(q_n(t))\,\mathrm{d}t = \int_0^1 \langle \nabla V(q_n(t)), q_n(t)\rangle\,\mathrm{d}t,$$

and when combined with (V'3) gives

$$\int_0^1 h_n - V(q_n(t))\,\mathrm{d}t \ge h_n - \frac{2h_n}{2+p} = \frac{h_n p}{2+p}.$$

Therefore

$$0 < \tau(q_n, q_n) = \frac{\mathcal{J}(h_n, q_n)}{\int_0^1 h_n - V(q_n(t))\,\mathrm{d}t} \le \frac{\beta_2(2+p)}{h_n p} < C,$$

for some $C > 0$ independent of n. The proof of Theorem 3.9 then shows that $\{q_n\}$ has a strongly convergent subsequence in X which by Lemma 3.1 yields a brake periodic orbit with energy h. □

3.7 Brake periodic orbits of specified energy

Under another hypothesis, appropriate to the case of bounded V and V with slow growth,

(V'4) $\quad\langle x, \nabla V(x)\rangle > 0$ for all $x \neq 0$, and for some $h^* \in (0, \mathcal{H}(V))$,

$$\liminf_{|x|\to\infty}\{2(V(x) - h^*) - \langle Sx, \nabla V(x)\rangle\} > 0,$$

we obtain the existence of a brake periodic orbit with energy h^*. Observation (c) of the Introduction is a special case of the following.

Theorem 3.18. *Suppose that V satisfies (V'1), (V'2) and (V'4). Then there exists a solution of (1.1) with energy h^*.*

Proof. Let $h_n \nearrow h^*$ be a sequence of energy levels on which the existence of solutions q_n of (1.1) is a consequence of Theorem 3.9. The strategy of the proof of the preceding theorem means that it suffices here to show that, for some $\alpha > 0$ and all n,

$$\int_0^1 h_n - V(q_n(t))\,\mathrm{d}t \geq \alpha.$$

Suppose that this is false and, without loss of generality, that

$$\int_0^1 h_n - V(q_n(t))\,\mathrm{d}t \to 0 \quad \text{as} \quad n \to \infty.$$

As in the proof of Lemma 3.16, the second part of (V$'$4) implies that $\{q_n\}$, and hence $\{V(q_n)\}$, is bounded in $L^\infty(0,1)$. Since

$$\int_0^1 \langle \nabla V(q_n(t)), q_n(t)\rangle\,\mathrm{d}t = 2\int_0^1 h_n - V(q_n(t))\,\mathrm{d}t \to 0,$$

and the integrand on the left is non-negative, by (V$'$4), it follows (for a subsequence) that $\langle \nabla V(q_n), q_n\rangle \to 0$ pointwise almost everywhere. By strict inequality in the first part of (V$'$4), $q_n \to 0$ pointwise almost everywhere and so, from the dominated convergence theorem,

$$2\int_0^1 h_n - V(q_n(t))\,\mathrm{d}t \to 2h^* > 0,$$

which is a contradiction. This completes the proof. $\square$

3.8 Existence extended to self-adjoint operators

Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be a linear invertible self-adjoint operator with at least one positive eigenvalue. The aim of this section is to find conditions on $V \in C^1(\mathbb{R}^N, [0,\infty))$ and $h > 0$ such that there exists a non-constant periodic function

w satisfying

$$Aw''(t) + \nabla V(w(t)) = 0, \tag{3.13a}$$

$$\frac{1}{2}\langle Aw'(t), w'(t)\rangle + V(w(t)) = h, \tag{3.13b}$$

$$w'(t_0) = w'(t_1) = 0. \tag{3.13c}$$

for all $t \in \mathbb{R}$, and for some $t_0 \neq t_1$. Since S is an invertible self-adjoint operator (3.13) is a clear generalisation of (1.1).

Denote the set of eigenvalues of A by

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m < 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n,$$

where $0 \leq m$, $1 \leq n$ and $n + m = N$.

Since A is invertible and self-adjoint, there exists P such that $P^* = P^{-1}$ and $A = (D^{1/2}P)^* S (D^{1/2}P)$, where

$$S = \begin{pmatrix} I_{n\times n} & 0_{m\times n} \\ 0_{n\times m} & -I_{m\times m} \end{pmatrix} \quad \text{and} \quad D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n, -\mu_1, \ldots, -\mu_m).$$

Let $C = D^{1/2}P$ then $C^{-1} = P^* D^{-1/2}$. Consider $\overline{V} : \mathbb{R}^N \to [0, \infty)$ defined by $\overline{V} = V \circ C^{-1}$.

If $\overline{V}$ satisfies (V$'$1) and (V$'$2), then by Theorem 3.9 there exists a solution u of (1.1) for all $h \in (0, \mathcal{H}(\overline{V}))$. Therefore

$$Su''(t) + \nabla \overline{V}(u(t)) = 0, \tag{3.14a}$$

$$\frac{1}{2}\langle Su'(t), u'(t)\rangle + \overline{V}(u(t)) = h, \tag{3.14b}$$

$$u'(t_0) = u'(t_1) = 0, \tag{3.14c}$$

for all $t \in \mathbb{R}$ and for some $t_0 \neq t_1$. Now, for all $x, y \in \mathbb{R}^N$,

$$\begin{aligned} \mathrm{d}\overline{V}[x](y) &= \mathrm{d}V[C^{-1}x]C^{-1}y = \langle \nabla V(C^{-1}x), C^{-1}y\rangle \\ &= \langle (C^{-1})^* \nabla V(C^{-1}x), y\rangle \\ &= \langle (C^*)^{-1} \nabla V(C^{-1}x), y\rangle. \end{aligned}$$

Hence for all $x \in \mathbb{R}^N$

$$\nabla \overline{V}(x) = (C^*)^{-1} \nabla V(C^{-1}x). \tag{3.15}$$

Let $w = C^{-1}u$ then by (3.14a) and (3.15),

$$\begin{aligned} Aw'' &= C^*SCw'' = C^*Su'' \\ &= -C^*\nabla\overline{V}(u) = C^*(C^*)^{-1}\nabla V(C^{-1}Cw) \\ &= -\nabla V(w), \end{aligned}$$

and also, by (3.14b) and (3.15),

$$\begin{aligned} h &= \overline{V}(u(t)) + \frac{1}{2}\nabla Su'(t), u'(t)\rangle \\ &= (V \circ C^{-1})Cw(t) + \frac{1}{2}\langle SCw'(t), Cw'(t)\rangle \\ &= V(w(t)) + \frac{1}{2}\langle Aw'(t), w'(t)\rangle. \end{aligned}$$

Finally, by (3.14c), $w'(t_0) = w'(t_1) = 0$. Hence w is a solution of (3.14) as required.

Note that by the definition of P and D there exists c_0, $c_1 > 0$ such that

$$c_0|x| \leq |P^*D^{-1}x| = |D^{-1/2}x| \leq c_1|x|,$$

for all $x \in \mathbb{R}^N$. Therefore $\mathcal{H}(\overline{V}) = \mathcal{H}(V)$. We have proved the following.

Theorem 3.19. *Suppose V satisfies (V′1) and $\overline{V}$ satisfies (V′2). Then there exists a solution of (3.13) for all $h \in (0, \mathcal{H}(V))$.*

Note that $\overline{V}$ satisfies (V′2) if, for example, ∇V is bounded.

3.9 Localisation principle

In this section we show that the results so far obtained may be applied to potentials with no restriction on the behaviour at infinity; instead information about the level sets of the potential is used.

Theorem 3.20. *Let $V \in C^1(\mathbb{R}^{n+m}, [0,\infty))$ satisfy $V(0) = 0$ and suppose*

(C) *there exists a bounded open convex set $\mathcal{C} \subset \mathbb{R}^{n+m}$ and $H > 0$ such that $0 \in \mathcal{C}$, $V(x) = H$ and $\nabla V(x) = 0$ for all $x \in \partial\mathcal{C}$.*

Then for almost all $h \in (0, H)$ there exists a solution u of (1.1) with $u(t) \in \overline{\mathcal{C}}$ for all $t \in \mathbb{R}$.

Proof. Define $V_e : \mathbb{R}^{n+m} \to [0,\infty)$ by

$$V_e(x) = \begin{cases} V(x) & x \in \mathcal{C}, \\ H & x \in \mathbb{R}^{n+m} \setminus \mathcal{C}. \end{cases} \tag{3.16}$$

Then by (C), $V_e \in C^1(\mathbb{R}^{n+m}, [0,\infty))$ and

$$\mathcal{H}(V_e) := \liminf_{|x|\to\infty} V_e(x) = H > 0.$$

Since $\mathcal{C}$ is relatively compact, ∇V_e is bounded. Hence (V$'$1) and (V$'$2) are satisfied, so by Theorem 3.9, for almost all $h \in (0, H)$ there exists a non-constant periodic function u satisfying

$$\begin{aligned} Su''(t) + \nabla V_e(u(t)) &= 0, \\ \frac{1}{2}\langle Su'(t), u'(t)\rangle + V_e(u(t)) &= h, \\ u'(t_0) = u'(t_1) &= 0. \end{aligned} \tag{3.17}$$

It remains to show that $u(t) \in \overline{\mathcal{C}}$ for all $t \in \mathbb{R}$. Note that $u'(t_0) = 0$ and $V_e(u(t_0)) = h$ hence $u(t_0) \in \mathcal{C}$. Suppose for a contradiction that the orbit of u leaves $\overline{\mathcal{C}}$. By (3.16) and (3.17), u has constant non-zero velocity outside of $\mathcal{C}$. Therefore u follows a straight line path in this region. Since $\mathcal{C}$ is convex, the orbit of u subsequently never intersects $\mathcal{C}$. This contradicts the periodicity of u. □

Chapter 4

Even Potentials of Indefinite Sign

Throughout this chapter, the potential V is assumed to be even and, in contrast to Chapter 3, allowed to have indefinite sign. Let $V \in C^1(\mathbb{R}^{n+m}, \mathbb{R})$. We seek solutions of (1.1). Due to the evenness of the potential we may assume the solution u is odd and that the boundary condition (1.1c) is of the form

$$u'(t_0) = 0 = u'(-t_0)$$

for some $t_0 > 0$. This restriction is one of the key observations which allow the existence theory of Chapter 2 and Chapter 3 to be refined.

4.1 Abstract theory refined

Let X be a separable Hilbert space and $X \neq Y \subset X$ a closed subspace so that $X = Y \oplus Z$ where $Z = Y^{\perp}$. For each $i \in \mathbb{N}_0$ let E_i be a finite dimensional subspace of X such that $E_i = (E_i \cap Y) \oplus (E_i \cap Z)$, $E_i \subset E_{i+1}$ and $\cup_{i \in \mathbb{N}_0} E_i$ is dense in X.

Let $0 < h_1 < h_2 < \infty$, $e \in E_1 \cap Z$ and $\|e\| = 1$. Let $\tau : X \times X \to \mathbb{R}$ be a continuous bilinear functional satisfying (T1), (T2), (T4) (see page 17) and

(T3*) $\quad 0 \leq -\tau(y, y) \leq c_0 \|y\|^2$ for all $y \in Y$.

Let $\mathcal{V} \in C^1(X, \mathbb{R})$ be such that $\mathcal{V}(0) = 0$. Put

$$\Lambda(h) = \{x \in X : \mathcal{V}(x) < h\}, \quad \Lambda_i(h) = \{x \in X : \mathcal{V}(x) < h\} \cap E_i$$

and define $J : \mathbb{R} \times X \to [0, \infty)$ by

$$J(h, x) = \big(\tau(x, x)\big)^{+}\big(h - \mathcal{V}(x)\big)^{+}.$$

Note that J is continuously differentiable at x whenever $J(h, x) > 0$. In the notation of Lemma 2.3, let P_i be the orthogonal projection of E_i onto $Y \cap E_i$. Suppose

(V0) $\mathcal{V}(y + \mu e) \geq 0 = \mathcal{V}(0)$ whenever $\mu \geq 0$ and $y \in Y$;

(V1) $\sup\{\mu > 0 : y + \mu e \in \Lambda(h_2),\, y \in Y\} = M < \infty$;

(V2*) $\{y + \mu e : y \in Y,\, \mu > 0\} \cap \Lambda_i(h_2)$ is bounded for each $i \in \mathbb{N}_0$;

(V3*) $P_i \Lambda_i(h_2)$ is bounded for each $i \in \mathbb{N}_0$.

Note that in general $P_i \Lambda_i(h_2) \neq Y \cap \Lambda_i(h_2)$. The following theorem is a refinement of Theorem 2.1. In the proof only steps not covered in the proof of Theorem 2.1 are detailed.

Theorem 4.1. *For almost all $h_0 \in [h_1, h_2]$ there is an increasing sequence $\{i_k\} \subset \mathbb{N}$, real numbers $\beta_2 > \beta_1 > 0$ and $C > 0$ such that there exists a sequence $\{x_k\}$ with $x_k \in E_{i_k}$ satisfying the following properties*

(i) $0 < \tau(x_k, x_k) \leq C$ *(independent of k);*

(ii) $0 < \beta_1 \leq J(h_0, x_k) \leq \beta_2$ *(independent of k and h_0);*

(iii) $2\tau(x_k, x)\big(h_0 - \mathcal{V}(x_k)\big) = \tau(x_k, x_k)\langle \nabla_{i_k} \mathcal{V}(x_k), x\rangle$ *for all $x \in E_{i_k}$;*

(iv) $x_k \in \Lambda_{i_k}(h_0)$.

Proof. As remarked earlier, hypothesis (V2*) is sufficient in the proof of Lemma 2.3. By careful bookkeeping it is apparent that the property $\mathcal{V} : \mathbb{R}^{n+m} \to [0, \infty)$ can be replaced by (V0) in the proof of Theorem 2.1 (with $\Omega = X$). So we begin our proof of Theorem 4.1 at the statement of inequality (2.11c). The next step in the proof of Theorem 2.1 requires that $\Lambda_k(h_0)$ is bounded so an alternative approach is now taken.

For $h \in (h_0, \overline{h}]$ let $\mathcal{A}(h, k)$ and $d_k(h)$ be as defined in (2.13) and (2.14) respectively. In the following we make crucial use of hypotheses (V3*) and (T3*).

Since $J(h_0, x) > 0$ for all $x \in \mathcal{A}(h,k)$, $\mathcal{A}(h,k) \subset \Lambda_k(h_0)$. Therefore $P_k(\mathcal{A}(h,k))$ is bounded by (V3*). By (T1),

$$\tau(x,x) = \tau(P_k x, P_k x) + \tau((I-P_k)x, (I-P_k)x) \quad \text{for all } x \in E_k.$$

Hence, by (T2) and (T3*),

$$\begin{aligned} c_0\|(I-P_k)x\|^2 &\le \tau((I-P_k)x, (I-P_k)x) \\ &= \tau(x,x) - \tau(P_k x, P_k x) \\ &\le \tau(x,x) + c_0\|P_k x\|^2. \end{aligned}$$

So, by definition of $\mathcal{A}(h,k)$ and the fact that $P_k(\mathcal{A}(h,k))$ is bounded, it follows that $(I-P_k)\mathcal{A}(h,k)$ is bounded. Hence, for $h \in (h_0, \overline{h}]$, $\mathcal{A}(h,k)$ is bounded and so compact.

Let $\mathcal{B}(\overline{h}, k)$ denote the set

$$\begin{aligned} \{x \in E_k :& \tau(x,x) \le \alpha + 4/k, \\ & c_k(h_0, \underline{h}) - (\overline{h} - h_0)/k \le J(h_0, x) \le c_k(h_0, \underline{h}) + (\alpha + 3/k)(\overline{h} - h_0)\}. \end{aligned}$$

Since $\mathcal{B}(\overline{h}, k)$ is a closed subset of $\mathcal{A}(\overline{h}, k)$ it is compact. Therefore there exists $\mu_k \in (0, 1/2]$, dependent on h_0, $\overline{h}$ and k, such that

$$|\tau(x_1, x_2) - \tau(x_2, x_2)| \le 1/k \quad \text{whenever} \quad x_1 \in \mathcal{B}(\overline{h}, k) \quad \text{and} \quad \|x_1 - x_2\| \le \mu_k. \tag{4.1}$$

Let ν_k be the pseudo-gradient vector field defined on page 23.

Lemma 4.2.

$$\min\{d_k(h), 1\} d_k(h) \le \frac{(\alpha + 4/k)(h - h_0)}{\mu_k}.$$

Proof. Suppose for a contradiction that the Lemma is false. Let $W : E_k \to E_k$ be defined by (2.15); the function W is well defined since $\mathcal{A}(h,k)$ is compact. Recall that W is locally Lipschitz continuous on E_k and $\|W(x)\| \le 2$ for all $x \in E_k$. Therefore the Cauchy problem

$$\dot{u}(t) = W(u(t)), \quad u(0) = x,$$

has a unique solution $u(t;x)$ defined for all t and $J(h_0, u(t;x))$ is a decreasing function of t for all x. Now define a homeomorphism U on E_k by

$$U(x) = u(\mu_k; x), \quad x \in E_k,$$

where μ_k is defined in (4.1). If $x \in \partial_k Q(\underline{h})$ then (2.6) and (2.9) imply that $x \in A_h$ and so $U(x) = x$. Therefore $U \circ \gamma \in \Gamma_k(\underline{h})$ for all $\gamma \in \Gamma_k(\underline{h})$. We consider in particular $U \circ \gamma$ when $\gamma \in \Gamma_k(\underline{h})$ satisfies the properties (2.11). Then there are two possibilities for $x \in Q_k(\underline{h})$.

In the first, $J(h_0, \gamma(x)) \le c_k(h_0, \underline{h}) - (h - h_0)/k$ and consequently

$$J(h_0, U \circ \gamma(x)) \le c_k(h_0, \underline{h}) - (h - h_0),$$

because $J(h_0, u(t; \gamma(x)))$ is decreasing in t.

In the second (2.11a), and hence (2.11b) and (2.11c), hold. For convenience let $\widehat{x} = \gamma(x)$ in this case. Then either

$$J(h_0, U(\widehat{x})) < c_k(h_0, \underline{h}) - (h - h_0)/k,$$

or, since $J(h_0, u(t; \widehat{x}))$ is decreasing,

$$J(h_0, u(t; \widehat{x})) \in [c_k(h_0, \underline{h}) - (h - h_0)/k, c_k(h_0, \underline{h}) + (\alpha + 3/k)(h - h_0)]$$

for all $t \in (0, \mu_k)$. Suppose we are in the latter case, that is $u(t; \widehat{x}) \in B_h$ for all $t \in (0, \mu_k)$. Then, by definition of J and (2.9b), $u(t; \widehat{x}) \in \Lambda(h_0)$ for all $t \in (0, \mu_k)$. Because of the choice of μ_k and the fact that $\|W\| \le 2$, it follows from (2.11b) that

$$\tau\big(u(t; \widehat{x}), u(t; \widehat{x})\big) \le \alpha + 5/k \text{ for all } t \in (0, \mu_k).$$

Hence $u(t; \widehat{x}) \in \mathcal{A}(h, k)$ for all $t \in (0, \mu_k)$ and so, by definition,

$$\begin{aligned}
J(h_0, U(\widehat{x})) &= J(h_0, \widehat{x}) - \int_0^{\mu_k} \langle \nu_k(u(t; \widehat{x}), \nabla_k J(h_0, u(t; \widehat{x}))\rangle \, \mathrm{d}t \\
&\le J(h_0, \widehat{x}) - \int_0^{\mu_k} \min\{\|\nabla_k J(h_0, u(t; \widehat{x}))\|, 1\} \|\nabla_k J(h_0, u(t; \widehat{x}))\| \, \mathrm{d}t \\
&\le J(h_0, \widehat{x}) - \mu_k \min\{d_k(h), 1\} d_k(h),
\end{aligned}$$

by the definition of the pseudo-gradient and the fact that $t \mapsto t\min\{t,1\}$ is increasing. Since we are supposing that the conclusion of the Lemma is false

$$\begin{aligned} J(h_0, U(\widehat{x})) &< c_k(h_0, \underline{h}) + (\alpha + 3/k)(h - h_0) - \mu_k\Big(\frac{(\alpha + 4/k)(h - h_0)}{\mu_k}\Big) \\ &= c_k(h_0, \underline{h}) - (h - h_0)/k. \end{aligned}$$

This shows that

$$\max_{x \in Q_k(\underline{h})} J(h_0, U \circ \gamma(x)) \le c_k(h_0, \underline{h}) - (h - h_0)/k.$$

Since $U \circ \gamma \in \Gamma_k(\underline{h})$, this contradicts the definition of $c_k(h_0, \underline{h})$, and the lemma is proven. □

Theorem 4.3. *For all $k \in \mathbb{N}$ there exist a critical point x_k of $\mathcal{J}(h_0, \cdot)$ in E_k with*

$$0 < \beta_1 \le c_k(h_0, \underline{h}) = \mathcal{J}(h_0, x_k) \le \beta_2; \tag{4.2a}$$

$$0 < \tau(x_k, x_k) \le \alpha + 5/k \quad (\alpha \textit{ independent of } k); \tag{4.2b}$$

$$2\,\tau(x_k, x)\big(h_0 - \mathcal{V}(x_k)\big) = \tau(x_k, x_k)\langle \nabla_k \mathcal{V}(x_k), x\rangle \textit{ for all } x \in E_k. \tag{4.2c}$$

Proof. The existence of a critical point of $\mathcal{J}(h_0, \cdot)$ on E_k with these properties is an immediate consequence of the preceding lemma, the fact that $\mathcal{J}(h, x) = J(h, x)$ whenever $J(h, x) > 0$, $\mathcal{J}(h_0, \cdot) \in C^1(X, \mathbb{R})$, the compactness of $\mathcal{A}(h, k)$ in the finite-dimensional space E_k and the estimates (4.2). □

Proof of Theorem 4.1. Since h_0 was chosen from a set of full measure in $[\underline{h}, \overline{h}]$, and since the right side of (2.2) is independent of $\underline{h}$, an arbitrary point of $[h_1, h_2)$, Theorem 4.1 follows from Theorem 4.3 with $C = \alpha + 5$. □

Theorem 4.4. *Let $\mathcal{V}$ satisfy (V0), (V1), (V2*), (V3*) and suppose τ satisfies (T1), (T2), (T3*) and (T4). If in addition (H) holds with $\Omega = X$ (see page 26) then for almost all $h \in [h_1, h_2]$ there exists $x \in \Lambda(h_0)$ with $J(h, x) > 0$ and $\nabla J(h, x) = 0$.*

Proof. The proof follows immediately from Theorem 4.1 and the proof of Theorem 2.7. □

4.2 Brake periodic orbits

In contrast to Chapter 3, we restrict the space X to odd periodic functions. When applying the abstract theory, definitions different to those of Chapter 3 are used. In particular, we change the definition of the Hilbert space X, the finite dimensional spaces E_i, the subspaces Y and Z, and the element e.

The refinement in Section 4.1, along with the assumption of evenness of V, allow a considerable relaxation of the conditions on V. Let

$$X = \{q \in W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^{n+m}) : q(1-t) = q(1+t),\, q(-t) = -q(t)\, \forall t \in \mathbb{R}\}$$

which is a Hilbert space when equipped with inner product

$$\langle q_1, q_2 \rangle = \int_0^1 \langle q_1'(t), q_2'(t) \rangle \mathrm{d}t.$$

The fact that X is a Hilbert space under the above inner product depends crucially on the fact that $q(0) = 0$ for all $q \in X$. Let $V \in C^1(\mathbb{R}^{n+m}, \mathbb{R})$ and define $\mathcal{V} : X \to \mathbb{R}$ and $\tau : X \times X \to \mathbb{R}$ as before on page 29.

Lemma 4.5. *For $h > 0$, let q be a critical point of $\mathcal{J}(h, \cdot)$ with $\mathcal{J}(h, q) > 0$. Let*

$$t_0^2 = \frac{\mathcal{J}(h, q)}{2(h - \int_0^1 V(q(t))\,\mathrm{d}t)^2} > 0.$$

Define $u : \mathbb{R} \to \mathbb{R}^{n+m}$ by $u(t) = q(t/t_0)$. Then u is twice continuously differentiable and satisfies (1.1).

Proof. The proof, which is similar to Lemma 3.1, is omitted. □

In order to apply Theorem 4.4 and Lemma 4.5 to J let

$$E_i = \operatorname{span}\{e_{j,k} : 0 \le k \le i,\, 1 \le j \le m+n\}, \quad i \in \mathbb{N},$$

where, for $1 \le j \le n+m$ and $k \in \mathbb{N}_0$,

$$e_{j,k}(t) := (0, \ldots, 0, \underbrace{\sin k\pi t}_{j\text{-th coeff.}}, 0, \ldots, 0) \in \mathbb{R}^{n+m}.$$

Remark 4.6. *With this choice of E_i, the conclusions of Lemmas 3.7 and 3.8 hold because E_i is a finite dimensional space of real analytic functions.*

For each $q \in X$ write $q(t) = (z(t), y(t))$ where $z(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^m$. Define

$$Y = \{(0, y) \in X\} \text{ and } Z = Y^{\perp} = \{(z, 0) \in X\}.$$

Let $\underline{e} \in \mathbb{R}^n$ satisfy $|\underline{e}| = 2$. Put $(e, 0)(t) = (\underline{e}, 0) \sin \pi t$ then $(e, 0) \in E_1 \cap Z$. Let

$$W = \{(\mu\underline{e}, c) : \mu \in \mathbb{R},\ c \in \mathbb{R}^m\} \subset R^{n+m}$$

and denote by P_W the orthogonal projection of $\mathbb{R}^{n+m}$ onto W. Now suppose $V \in C^1(\mathbb{R}^{n+m}, \mathbb{R})$ is even. Let

$$\mathcal{H}_W(V) = \liminf_{|P_W x| \to \infty} V(x)$$

and suppose the potential V satisfies the following.

(V$'$0) $V(0) = 0 \leq V(x)$ for all $x \in W$;

(V$'$1*) $0 < \mathcal{H}_W(V)$;

(V$'$2) there exist K, M, $G \geq 0$ and $0 < \gamma < 2$ such that

$$\langle z, \partial_z V(z, y) \rangle \leq K + MV(x) + G|x|^{\gamma}$$

for all $x = (z, y) \in \mathbb{R}^{n+m}$.

Condition (V$'$1*) determines the distinction between n and m in the definition of S. This allows us to distinguish between positive energy solutions and negative energy solutions. In what follows only positive energy brake periodic orbits are found.

The next theorem is a refinement of Theorem 3.9 in the case when V is even.

Theorem 4.7. *Suppose V is even and satisfies hypotheses (V$'$0), (V$'$1*) and (V$'$2). Then there exists a solution of (1.1) for almost all $h \in (0, \mathcal{H}_W(V))$.*

Proof. By Theorem 4.4 and Lemma 4.5 it suffices to show that (T1), (T2), (T3*), (T4), (V0), (V1), (V2*), (V3*) and (H) hold.

The properties (T1) and (T4) follow trivially. The properties (T2) and (T3*) follow by Wirtinger's inequality and the definition of S, Y and Z. Note in particular that $q(0) = 0$ for all $q \in X$.

The property (V0) follows immediately from (V'0) since $\mu(e(t), 0) + (0, y(t)) \in W$ for all $t \in \mathbb{R}$, $\mu \geq 0$ and $(0, y) \in Y$.

Let $h \in (0, \mathcal{H}_W(V))$ be arbitrary but fixed. To show (V1) holds, define $g_W : [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ by

$$g_W(\rho) = \inf_{|P_W x| \geq \rho} V(x).$$

Then g_W is increasing and $\lim_{\rho \to \infty} g_W(\rho) = \mathcal{H}_W(V)$. Suppose $(\mu e, y) \in \Lambda(h)$ where $\mu > 0$ and $(0, y) \in Y$. Then by definition of $\Lambda(h)$

$$\begin{aligned} h &> \int_0^1 V(\mu e, y)\, \mathrm{d}t \geq \int_0^1 g_W(|(\mu e, y)|)\, \mathrm{d}t \\ &\geq \int_0^1 g_W(2|\mu| \sin \pi t)\, \mathrm{d}t. \end{aligned}$$

Hence $\sup\{\mu > 0 : (\mu e, y) \in \Lambda(h),\ (0, y) \in Y\} < \infty$ and so (V1) holds.

To verify (V2*) suppose there exists $\{q_k\} \subset \{\mu(e, 0) + (0, y) \in \Lambda_i(h) : (0, y) \in Y\}$, for fixed $i \in \mathbb{N}$, such that $\|q_k\| \to \infty$ as $k \to \infty$. By Remark 4.6 there exists an increasing sequence $\{k_j\} \subset \mathbb{N}$ and $U \subset [0, 1]$ with $\operatorname{meas} U > h/\mathcal{H}_W(V)$ such that $|q_{k_j}(t)| \to \infty$ uniformly for $t \in U$. Then, since $q_k : \mathbb{R} \to W$,

$$h > \int_0^1 V(q_{k_j}(t))\, \mathrm{d}t \geq \int_U g_W(|q_{k_j}(t)|)\, \mathrm{d}t \to \mathcal{H}_W(V) \operatorname{meas} U$$

for as $j \to \infty$. This contradicts the fact that $\operatorname{meas} U > h/\mathcal{H}_W(V)$ and so completes the proof of (V2*).

The verification of (V3*) follows similarly: suppose, for fixed $i \in \mathbb{N}$, there exists $\{(z_k, y_k)\} \subset \{(z, y) \in \Lambda_i(h) : (z, y) \in Z \times Y\}$ such that $\|P_i(z_k, y_k)\| = \|(0, y_k)\| \to \infty$ as $k \to \infty$. By Remark 4.6 there exists an increasing sequence $\{k_j\} \subset \mathbb{N}$ and $U \subset [0, 1]$ with $\operatorname{meas} U > h/\mathcal{H}_W(V)$ such that $|(0, y_{k_j}(t))| \to \infty$

uniformly for $t \in U$. Then

$$h > \int_0^1 V((z_{k_j}, y_{k_j}(t)))\,\mathrm{d}t \geq \int_U g_W(|(0, y_{k_j}(t))|)\,\mathrm{d}t \to \mathcal{H}_W(V)\mathrm{meas}\,U$$

for as $j \to \infty$. This contradicts the fact that $\mathrm{meas}\,U > h/\mathcal{H}_W(V)$ and so completes the proof of (V3*).

To prove (H1), let $\gamma_2 > \gamma_1 > 0$, $D > 0$ be arbitrary and fixed. Recall the set

$$R = \cup_{i \in \mathbb{N}}\{q \in \Lambda_i(h) : \nabla_i \mathcal{J}(h,q) = 0, \gamma_1 \leq \mathcal{J}(h,q) \leq \gamma_2, 0 < \tau(q,q) \leq D\}.$$

Let $q \in R$, then (3.8) in the verification of (H1) from the proof of Theorem 3.9, states

$$\int_0^1 |q'(t)|^2\,\mathrm{d}t < \frac{D^2}{\gamma_1}\left(K + Mh + G\left(\int_0^1 |q(t)|^2\,\mathrm{d}t\right)^{\frac{\bar{\gamma}}{2}}\right).$$

This assumes (V$'$2) but neither (V$'$1), (V$'$1*) nor $V \geq 0$. Since $q(0) = 0$,

$$\int_0^1 |q'(t)|^2\,\mathrm{d}t < \frac{D^2}{\gamma_1}\left(K + Mh + G\left(\int_0^1 |q'(t)|^2\,\mathrm{d}t\right)^{\frac{\bar{\gamma}}{2}}\right).$$

Hence R is bounded in X and so (H1) holds. In the following we use the fact that X is a subspace of $W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^{n+m})$. The condition (H2) follows exactly as in the proof of Theorem 3.9. Finally (H3) also holds since the proof of Theorem 3.9 shows if $\{q_k\} \subset R$ with $\nabla \mathcal{J}(h, q_k) \to 0$ in X^* then, for a subsequence, $q_k \rightharpoonup q$ and $\|q_k'\|_{L^2} \to \|q'\|_{L^2}$ for some $q \in X$. □

4.3 Dimensional restrictions and non-triviality

Assumptions (V$'$0) and (V$'$1*) refer to the behaviour of the potential V on the space W. It is therefore natural to ask whether the brake periodic orbits found in Theorem 4.7 have range contained in W. The following is a simple condition on V that ensures all brake periodic orbits do not solely lie in W.

$$(W) \quad \text{If} \quad (I - P_W)\nabla V(P_W x) = 0 \quad \text{then} \quad P_W x = 0.$$

We show that (W) is incompatible with evenness of V whenever $n < m+2$. This implies, when seeking an example of an even potential V satisfying (W), we must assume $n \geq m+2$.

Theorem 4.8. *Let $F : \mathbb{R}^N \to \mathbb{R}^M$ be odd with $M < N$. Then there exists $x \in \mathbb{R}^N \setminus \{0\}$ such that $F(x) = 0$.*

Proof. This is a straightforward consequence of [Llo78, Thm 3.2.7, p45]. □

Corollary 4.9. *Whenever $n < m+2$ there exists $P_W x \neq 0$ such that $(I - P_W)\nabla V(P_W x) = 0$.*

Proof. Since V is even, ∇V is odd. Therefore $F : P_W\mathbb{R}^{n+m} \to (I - P_W)\mathbb{R}^{n+m}$ defined by

$$F(x) = (I - P_W)\nabla V(x) \quad \text{for all } x \in P_W\mathbb{R}^{n+m}$$

is an odd mapping. Also, since V is continuously differentiable, F is continuous. If $n < m+2$ then

$$n - 1 = \dim(I - P_W)\mathbb{R}^{n+m} < \dim P_W\mathbb{R}^{n+m} = m+1.$$

Hence Theorem 4.8 applies to complete the proof. □

Example 4.10. *Consider the case of $n = 3$ and $m = 1$, so that $n = m+2$. Let $V \in C^1(\mathbb{R}^{3+1}, \mathbb{R})$ be defined by*

$$V(a,b,c,d) = f(ab) + g(cd) + a^2 + d^2 \quad \textit{for all } (a,b,c,d) \in \mathbb{R}^{3+1},$$

where $f, g \in C^1(\mathbb{R}, \mathbb{R})$ satisfy

$$\begin{aligned} &f(0) = 0 = g(0), \quad f'(0) \neq 0 \neq g'(0), \quad \sup\{|g(s)| : s \in \mathbb{R}\} < \infty \quad \textit{and} \\ &sf'(s) \leq C(1 + f(s)) \quad \textit{for all } s \in \mathbb{R}, \end{aligned}$$

where $C > 0$. In the case $n = 3$ and $m = 1$, $W = \{(a,0,0,d) : a, d \in \mathbb{R}\}$, so that

$$\begin{aligned} (I - P_W)\nabla V(P_W(a,b,c,d)) &= (0, af'(0), dg'(0), 0) \\ &= 0 \quad \textit{if and only if} \quad P_W(a,b,c,d) = 0. \end{aligned}$$

The conditions (V0), (V1), (V2) and (V3*) also hold.*

4.4 Brake periodic orbits for all positive energies

This section complements Section 3.6. Theorem 3.12 exhibits an *even* potential for which there are no brake periodic orbits with energies in a prescribed set of zero measure. As in Theorem 3.17, in order to prove existence for all positive energies, stronger hypotheses than those assumed in the previous section must be used.

(V′3*) There exists $p > 0$ such that $\langle \nabla V(x), P_W x\rangle \geq pV(x)$ for all $x \in \mathbb{R}^{n+m}$ and $V(x) > 0$ for all $x \in W \setminus \{0\}$.

Theorem 4.11. *Suppose that the even potential V satisfies (V′2) and (V′3*). Then there exists a solution of* (1.1) *for all $h > 0$.*

Proof. Note that (V′3*) and (V′0) ensure $\liminf_{|P_W x|\to\infty} V(x) = \infty$ so that (V′1) holds with $\mathcal{H}_W(V) = \infty$. Choose and fix $h \in (0, \mathcal{H}_W(V))$. Suppose $h_n \to h$ as $n \to \infty$ and $\{q_n\} \subset \Lambda(h)$ satisfy $h_n \in (0, \mathcal{H}_W(V))$,

$$0 < \inf_n J(h_n, q_n) \leq \sup_n J(h_n, q_n) < \infty \quad \text{and} \quad \nabla J(h_n, q_n) = 0.$$

By the same argument as used in the proof of Theorem 3.17, it is sufficient to prove

$$\inf_n \int_0^1 h_n - V(q_n(t))\,\mathrm{d}t > 0.$$

Since $\langle \nabla J(h_n, q_n), P_w q_n\rangle = 0$

$$\begin{aligned} 2\int_0^1 h_n - V(q_n(t))\,\mathrm{d}t \int_0^1 \langle Sq_n'(t), P_W q_n'(t)\rangle\,\mathrm{d}t \\ = \int_0^1 \langle \nabla V(q_n(t)), P_W q_n(t)\rangle\,\mathrm{d}t \int_0^1 \langle Sq_n'(t), q_n'(t)\rangle\,\mathrm{d}t. \end{aligned} \tag{4.3}$$

Equality (4.3), hypothesis (V′3*) and the fact that $J(h_n, q_n) > 0$, $q_n \in \Lambda(h)$ and $\langle Sx, P_W x\rangle \leq \langle Sx, x\rangle$ for all $x \in \mathbb{R}^{n+m}$, imply

$$2\int_0^1 h_n - V(q_n(t))\,\mathrm{d}t \geq \int_0^1 \langle \nabla V(q_n(t)), P_W q_n(t)\rangle\,\mathrm{d}t. \tag{4.4}$$

Hence by hypothesis (V$'$3*)

$$2\int_0^1 h_n - V(q_n(t))\,\mathrm{d}t \geq p\int_0^1 V(q_n(t))\,\mathrm{d}t.$$

Therefore

$$\int_0^1 h_n - V(q_n(t))\,\mathrm{d}t \geq \frac{h_n p}{2+p},$$

which completes the proof. $\square$

4.5 Brake periodic orbits of specified energy

This section uses arguments as in Section 3.7. In contrast to Theorem 3.18 there is no *a priori* bound on brake periodic orbits of even potentials of indefinite sign as the potential's level sets maybe unbounded. The additional assumption that the potential is bounded overcomes this difficulty. Under a new hypothesis

(V$'$4*) $\langle P_W x, \nabla V(x)\rangle > 0$ for all $x \neq 0$, and $|V|$ is bounded,

we obtain the existence of a brake orbit with energy h^* for all $h^* \in (0, \mathcal{H}_W(V))$.

Theorem 4.12. *Suppose that V satisfies (V$'$0), (V$'$1*), (V$'$2) and (V$'$4*). Then there exists a solution of (1.1) with energy h^* for all $h^* \in (0, \mathcal{H}_W(V))$.*

Proof. The proof follows that of Theorem 3.18; the details of this proof are included for clarity. Let $h_n \nearrow h^*$ be a sequence of energy levels on which the existence of solutions of (1.1) is a consequence of Theorem 4.7. The strategy of the proof of the preceding theorem means that it suffices here to show that, for some $\alpha > 0$ and all n,

$$\int_0^1 h_n - V(q_n(t))\,\mathrm{d}t \geq \alpha.$$

Suppose that this is false and, without loss of generality, that

$$\int_0^1 h_n - V(q_n(t))\,\mathrm{d}t \to 0 \quad \text{as} \quad n \to \infty.$$

Since, by (4.4),

$$\int_0^1 \langle \nabla V(q_n(t)), P_W q_n(t)\rangle\,\mathrm{d}t \leq 2\int_0^1 h_n - V(q_n(t))\,\mathrm{d}t \to 0,$$

and the integrand on the left is non-negative, by (V′4*), it follows (for a subsequence) that $\langle \nabla V(q_n), P_W q_n \rangle \to 0$ pointwise almost everywhere. The potential V is bounded and by strict inequality in the first part of (V′4*), $q_n \to 0$ pointwise almost everywhere and so, from the dominated convergence theorem,

$$2 \int_0^1 h_n - V(q_n(t))\, \mathrm{d}t \to 2h^* > 0,$$

which is a contradiction. This completes the proof. □

Chapter 5

Potentials with Blow-up

In this section we consider potentials which blow-up on the boundary of their domain of definition. Let $V \in C^1(\mathcal{O}, [0, \infty))$ where $\mathcal{O}$ is a bounded open subset of $\mathbb{R}^{n+m}$ with $0 \in \mathcal{O}$, and suppose that

$$\lim_{x \to \partial\mathcal{O}} V(x) = \infty.$$

We refer to such potentials as singular potentials. With further assumptions on V, we establish, for almost all prescribed $h > 0$, the existence of a brake periodic orbit u of (1.1). The solution u of (1.1) satisfies $u(t) \in \mathcal{O}$ for all $t \in \mathbb{R}$ and, since the orbit of u is compact, it is bounded away from $\partial\mathcal{O}$ and hence away from the singularities of V. The fact that the operator S can be indefinite makes the dealing with singularities non-trivial.

The problem of establishing existence of brake periodic orbits for singular potentials is straightforward in the positive definite case of $S = I$. Let V be a singular potential. Suppose $\overline{\Lambda(h)} = \{x \in \mathcal{O} : V(x) \leq h\}$ is connected and $\langle \nabla V(x), x \rangle > 0$ whenever $V(x) = h$. Fix $H > h$ and choose $A \subset \mathbb{R}^{n+m}$ such that A is closed, $\overline{\mathbb{R}^{n+m} \setminus A} \subset \mathcal{O}$ and $\overline{\Lambda(h)} \cap A = \emptyset$. Choose $\varepsilon > 0$ such that $\overline{\Lambda(h)}_\varepsilon := \{x : |x - y| \leq \varepsilon \, \forall y \in \overline{\Lambda(h)}\}$ satisfies $\overline{\Lambda(h)}_\varepsilon \cap A = \emptyset$. Let f be a continuous function on $\mathbb{R}^{n+m}$ satisfying

$$f(x) = \begin{cases} 0 \;:\; x \in \overline{\Lambda(h)} \\ 1 \;:\; x \in A \end{cases} \quad \text{and} \quad 0 \leq f(x) \leq 1 \text{ for all } x \in \mathbb{R}^{n+m}.$$

Let f_ε be the $\varepsilon/2$ mollification of f. Then f_ε is smooth and $f_\varepsilon(x) = 0$ whenever

$x \in \overline{\Lambda(h)}$. Finally let

$$\widetilde{V}(x) = Hf_\varepsilon(x) + V(x)(1 - f_\varepsilon(x)) \text{ for all } x \in \mathbb{R}^{n+m}.$$

Then $\widetilde{V} \in C^1(\mathbb{R}^{n+m}, [0, \infty))$ and satisfies $\liminf_{|x| \to \infty} \widetilde{V}(x) = H$ and

$$\widetilde{V}(x) = V(x) \text{ if } x \in \mathcal{O} \text{ and } V(x) \leq h, \quad h \leq \widetilde{V}(x) \leq H \text{ otherwise.}$$

The existence theory of Chapter 3 then applies to $\widetilde{V}$. Since in this case S is positive definite, the resulting brake periodic orbit u satisfies $\widetilde{V}(u(t)) \leq h$ for all $t \in \mathbb{R}$ and hence is a solution of (1.1).

5.1 Brake periodic orbits

As in Chapter 3 let

$$X = \{q \in W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^{n+m}) : q(t) = q(2+t) \text{ and } q(-t) = q(t) \, \forall t \in \mathbb{R}\}$$

but now put

$$\Omega = \{q \in X : q(t) \in \mathcal{O} \; \forall t \in \mathbb{R}\},$$

which is clearly bounded in L^∞ since $\mathcal{O}$ is bounded in $\mathbb{R}^{n+m}$. Then Ω is an open subset of X with $0 \in \Omega$. Since $V \in C^1(\mathcal{O}, [0, \infty))$ the functional $\mathcal{V} : \Omega \to [0, \infty)$ defined by

$$\mathcal{V}(q) = \int_0^1 V(q(t)) \, \mathrm{d}t$$

for all $q \in \Omega$ lies in $C^1(\Omega, [0, \infty))$. The continuous bilinear functional $\tau : X \times X \to \mathbb{R}$, the element $(e, 0)$ and the subspaces E_i, Y and Z of X are defined as in Chapter 3. We now show that the abstract theory developed in Section 2 applies to yield solutions of (1.1).

The following condition is a slight modification of the 'strong force condition' introduced by Gordon [Gor75]; it was originally posed for potentials whose singularities lie at the origin.

Definition 5.1 (Strong Force Condition). *Let $V \in C^1(\mathcal{O}, [0, \infty))$ where $\mathcal{O}$ is an open subset of $\mathbb{R}^{n+m}$ with $0 \in \mathcal{O}$. Then potential V is said to satisfy the* *'strong force condition', denoted (SF), if*

(i) $\lim_{x\to\partial\mathcal{O}} V(x) = \infty$,

and there exists an open neighbourhood $\mathcal{N}$ *of* $\partial\mathcal{O}$ *and a function* $U \in C^1(\mathcal{N}\cap\mathcal{O},[0,\infty))$ *satisfying*

(ii) $\lim_{x\to\partial\mathcal{O}} U(x) = \infty$,

(iii) $V(x) \geq |\nabla U(x)|^2$ *for all* $x \in \mathcal{N}\cap\mathcal{O}$.

Lemma 5.2. [Gor75] [Gre88, Lemma 2.1] *Suppose* $V \in C^1(\mathcal{O},[0,\infty))$ *satisfies (SF). If* $\{q_n\} \subset \Omega$ *and* $q_n \to q \in X \setminus \Omega$ *weakly and uniformly, then*

$$\lim_{n\to\infty} \int_0^1 V(q_n(t))\,\mathrm{d}t = \infty.$$

Proof. [Gre88, Lemma 2.1] If $q_n(t) \to \partial\mathcal{O}$ uniformly on a set of non-zero measure the result is immediate. So, suppose there exists $t_1, t_2 \in \mathbb{R}$ such that $q_n(t_1) \to q(t_1) \in \mathcal{N}\cap\mathcal{O}$, $q_n(t_2) \to q(t_2) \in \partial\mathcal{O}$ and $q(t) \in \mathcal{N}\cap\mathcal{O}$ for all $t \in (t_1,t_2)$. Then

$$\begin{aligned}
U(q_n(t_2)) - U(q_n(t_1)) &= \int_{t_1}^{t_2} \langle \nabla U(q_n(t)), q_n'(t)\rangle\,\mathrm{d}t \\
&\leq \left(\int_{t_1}^{t_2} |\nabla U(q_n(t))|^2\,\mathrm{d}t\right)^{1/2} \left(\int_{t_1}^{t_2} |q_n'(t)|^2\,\mathrm{d}t\right)^{1/2} \\
&\leq \sup_{m\in\mathbb{N}} \|q_m\| \left(\int_0^1 V(q_n(t))\,\mathrm{d}t\right)^{1/2},
\end{aligned}$$

and so, by the definition of U,

$$\int_0^1 V(q_n(t))\,\mathrm{d}t \to \infty \quad \text{as } n\to\infty.$$

□

Suppose in addition to (SF) that V satisfies the following condition,

(B2) there exists $B \geq 1$ such that $\langle \nabla V(x), Sx\rangle \leq B\langle \nabla V(x), x\rangle$ for all $x \in \mathcal{N}\cap\mathcal{O}$.

The set $\mathcal{N}$ in (SF) and (B2) can, without loss of generality, be chosen to be the same set. The condition (B2) is analogous to condition (B1) on page 40.

Theorem 5.3. *Suppose V satisfies (SF) and (B2). Then there exists a solution of (1.1) for almost all $h > 0$.*

Proof. We show the conditions of Theorem 2.8 hold. Let $0 < h_1 < h_2$ be arbitrary and fixed. The hypothesis (T) (page 17) has been verified in the proof of Theorem 3.9. Conditions (V1) and (V2) (page 16) follow by Lemma 3.8, the fact that $\lim_{x\to\partial\mathcal{O}} V(x) = \infty$ and the same argument as in the proof of Theorem 3.9.

To show (V3) suppose $\{q_k\} \subset \Lambda(h_2)$ and $q_k \to q$ strongly in X as $k \to \infty$. By taking a subsequence we may suppose $q_k \to q$ weakly and uniformly as $k \to \infty$. Suppose for a contradiction that $q \notin \Omega$. Then by (SF) and Lemma 5.2

$$\lim_{k\to\infty} \int_0^1 V(q_k(t))\,\mathrm{d}t = \infty,$$

which contradicts the fact that $q_k \in \Lambda(h_2)$ for all $k \in \mathbb{N}$.

Now we verify (H1). Let $\gamma_2 > \gamma_1 > 0$, $D > 0$ and $h \in [h_1, h_2]$ be arbitrary and fixed. Consider the set

$$R = \cup_{i\in\mathbb{N}}\{q \in \Lambda_i(h) : \nabla_i \mathcal{J}(h,q) = 0, \gamma_1 \le \mathcal{J}(h,q) \le \gamma_2, 0 < \tau(q,q) \le D\}.$$

We wish to show that R is bounded in X. Since R is bounded in L^∞ it is sufficient to show that $\sup_{q\in R} \|q'\|_{L^2} < \infty$. Let $q \in R$, then

$$\tau(q,q) = \int_0^1 \langle Sq'(t), q'(t)\rangle\,\mathrm{d}t \le D \tag{5.1}$$

and

$$\gamma_1 \le \int_0^1 \langle Sq'(t), q'(t)\rangle\,\mathrm{d}t \int_0^1 h - V(q(t))\,\mathrm{d}t \le \gamma_2. \tag{5.2}$$

Now $\langle \nabla\mathcal{J}(h,q), q\rangle = 0$, $\mathcal{J}(h,q) \ge \gamma_1 > 0$ and $q \in \Lambda(h)$ imply

$$\int_0^1 \langle \nabla V(q(t)), q(t)\rangle\,\mathrm{d}t = 2\int_0^1 h - V(q(t))\,\mathrm{d}t < 2h. \tag{5.3}$$

Since $S(E_i) \subset E_i$ for all $i \in \mathbb{N}_0$, $\langle \mathcal{J}(h,q), Sq\rangle = 0$ and so, since $\mathcal{J}(h,q) > 0$,

$$\int_0^1 |q'(t)|^2\,\mathrm{d}t = \frac{\int_0^1 \langle Sq'(t), q'(t)\rangle\,\mathrm{d}t \int_0^1 \langle \nabla V(q(t)), S(q(t))\rangle\,\mathrm{d}t}{2\int_0^1 h - V(q(t))\,\mathrm{d}t}.$$

Since $\mathcal{O} \setminus \mathcal{N}$ is a closed subset of the bounded set $\mathcal{O}$,

$$C := \max\{\langle \nabla V(x), Sx \rangle, -\langle \nabla V(x), x \rangle : x \in \mathcal{O} \setminus \mathcal{N}\} < \infty. \tag{5.4}$$

Therefore, by (5.1) and (5.2),

$$\begin{aligned}
&\int_0^1 |q'(t)|^2 \, \mathrm{d}t \le \frac{D^2}{2\gamma_1} \int_0^1 \langle \nabla V(q(t)), Sq(t) \rangle \, \mathrm{d}t \\
&= \frac{D^2}{2\gamma_1} \left\{ \int_{\{t : q(t) \in \mathcal{O} \setminus \mathcal{N}\}} \langle \nabla V(q(t)), Sq(t) \rangle \, \mathrm{d}t + \int_{\{t : q(t) \in \mathcal{N} \cap \mathcal{O}\}} \langle \nabla V(q(t)), Sq(t) \rangle \, \mathrm{d}t \right\} \\
&\le \frac{D^2}{2\gamma_1} \left\{ C + B \int_{\{t : q(t) \in \mathcal{N} \cap \mathcal{O}\}} \langle \nabla V(q(t)), q(t) \rangle \, \mathrm{d}t \right\} \quad \text{by (5.4) and (B2),} \\
&= \frac{D^2}{2\gamma_1} \left\{ C + B \left(\int_0^1 \langle \nabla V(q(t)), q(t) \rangle \, \mathrm{d}t - \int_{\{t : q(t) \in \mathcal{O} \setminus \mathcal{N}\}} \langle \nabla V(q(t)), q(t) \rangle \, \mathrm{d}t \right) \right\} \\
&\le \frac{D^2}{2\gamma_1} \{ C + B\,(2h + C) \} \quad \text{by (5.3) and (5.4).}
\end{aligned}$$

This completes the verification of (H1). Since $\overline{\Lambda(h)} \subset \Omega$ the conditions (H2) and (H3) follow exactly as in the proof of Theorem 3.9. We have shown all the conditions of Theorem 2.8 are satisfied. Therefore there exists β_1, $\beta_2 \in \mathbb{R}$ such that for almost all $h \in [h_1, h_2]$ there exists $q \in \Lambda(h)$ satisfying

$$\nabla \mathcal{J}(h, q) = 0 \quad \text{and} \quad 0 < \beta_1 \le \mathcal{J}(h, q) \le \beta_2, \tag{5.5}$$

where β_1 and β_2 are independent of $h \in [h_1, h_2]$. Then by Lemma 3.1 there exists a non-constant solution of (1.1). Since h_1 and h_2 were chosen arbitrarily in $(0, \infty)$ the proof is complete. □

The compatibility of the conditions in Theorem 5.3 is established by the following example.

Example 5.4. *Let* $\mathcal{O} = \{x \in \mathbb{R}^2 : |x| < 1\}$ *and* $\varepsilon > 0$. *Define* $U, V \in C^1(\mathcal{O} \setminus \{0\}, \mathbb{R}^2)$ *by*

$$U(x) = \frac{1}{4\varepsilon(1 - |x|^2)^{\varepsilon}} \quad \textit{and} \quad V(x) = \frac{|x|^2}{(1 - |x|^2)^{2(1+\varepsilon)}} - 1 \quad \textit{for all } x \in \mathcal{O}.$$

Then

$$|\nabla U(x)|^2 = \frac{1}{4(1-|x|^2)^{2(1+\varepsilon)}} \leq V(x) \quad \textit{for } 1-|x|^2 \textit{ sufficiently small.}$$

Clearly $U(x), V(x) \to \infty$ *as* $x \to \partial\mathcal{O}$. *Therefore* V *satisfies (SF). The condition (B) is satisfied trivially with* $B = 1$.

Chapter 6

Extension to Infinite Dimensions

Denote the separable Hilbert space of real square summable sequences by l_2. In this chapter we consider the problem of existence of solutions $u : \mathbb{R} \to l_2$ of

$$Su''(t) + \nabla V(u(t)) = 0, \tag{6.1a}$$

$$\frac{1}{2}\langle Su'(t), u'(t)\rangle_{l_2} + V(u(t)) = h^*, \tag{6.1b}$$

for all $t \in \mathbb{R}$, where $V \in C^1(l_2, [0, \infty))$ and $S : l_2 \to l_2$ is given by $(Sx)_i = s_i x_i$ for all $i \in \mathbb{N}$ with

$$s_i = \begin{cases} +1 & \text{if } 1 \leq i \leq p, \\ -1 & \text{if } p < i, \end{cases}$$

for some fixed $p \in \mathbb{N}$. Here $h^* > 0$ is called, as before, the energy and, for a brake periodic orbit, u is non-constant periodic and there exists some $t_0 \neq t_1$ such that

$$u'(t_0) = u'(t_1) = 0. \tag{6.1c}$$

The notation $u'(t)$ is used to mean $u'(t) \in l_2$ with $u'(t)s = \mathrm{d}u[t](s)$ for all $s \in \mathbb{R}$. Similarly, $u''(t)$ is the element in l_2 such that $u''(t)s_1s_2 = \mathrm{d}u'[t](s_1)(s_2)$ for all $s_1, s_2 \in \mathbb{R}$.

In contrast to Chapter 3, we find that we cannot prove existence for almost all prescribed energies h^* in a given interval. However, under assumptions on V, we prove for each $h > 0$ there is a non-constant solution of (6.1) with $h^* \in (0, h]$. A simple consequence of this is that there exists a countable number of solutions whose energies accumulate at zero.

6.1 Galerkin theory of critical points revisited

We follow closely the abstract theory developed in Chapter 2, highlighting differences carefully.

Let X be a separable Hilbert space and $X \neq Y \subset X$ be a closed subspace so that $X = Y \oplus Z$ where $Z = Y^{\perp}$. Let E_i be subspaces of X, $i \in \mathbb{N}$. We do not assume that the E_i are finite dimensional. Let $P_N : X \to X$ be such that

(i) $P_N : X \to X$ is a projection for all $N \in \mathbb{N}$,

(ii) $P_N E_i = (P_N E_i \cap Y) \oplus (P_N E_i \cap Z)$ for all $i, N \in \mathbb{N}$,

(iii) $E_i \subset E_{i+1}$ for all $i \in \mathbb{N}$,

(iv) $\bigcup_{i\in\mathbb{N}} P_N E_i$ is dense in $P_N X$, and

(v) $P_N E_i$ is finite dimensional for all $i, N \in \mathbb{N}$.

By (i) it follows that $P_N X$ is closed in X. In the following we pay particular attention to dependencies on N. Let $\nabla u(x)$ denote the gradient at $x \in X$, with respect to the inner product $\langle \cdot, \cdot \rangle$ in X, of a C^1 functional $u : X \to \mathbb{R}$. If $x \in P_N E_i$ let $\nabla_i^N u(x) \in P_N E_i$ denote the gradient of its restriction to $P_N E_i$, and if $x \in P_N X$ let $\nabla^N u(x) \in P_N X$ denote the gradient of its restriction to $P_N X$ with respect to the same inner product.

Let $0 < h_2 < \infty$, $e \in P_1 E_1 \cap Z$, $\|e\| = 1$, $\mathcal{V} : X \to [0, \infty)$ be a functional and for any $h \in [0, \infty)$ denote

$$\Lambda(h) = \{x \in X : \mathcal{V}(x) < h\}.$$

Suppose the following properties hold:

($\mathfrak{V}0$) $\mathcal{V} \in C^1(X, [0, \infty))$ and $\mathcal{V}(0) = 0$,

($\mathfrak{V}1$) $\sup\{\mu > 0 : y + \mu e \in \Lambda(h_2),\, y \in Y\} = M < \infty$,

($\mathfrak{V}2$) $\Lambda_i^N(h_2) := \Lambda(h_2) \cap P_N E_i$ is bounded in X for all $i, N \in \mathbb{N}$.

Suppose that $\tau : X \times X \to \mathbb{R}$ is a continuous, symmetric, bilinear functional satisfying conditions (T1)–(T4) (see page 17). For all $h \in (0, h_2)$ and $x \in X$, define

$$\mathcal{J}(h,x) = \tau(x,x)\big(h - \mathcal{V}(x)\big) \quad \text{and} \quad J(h,x) = (\tau(x,x))^+\big(h - \mathcal{V}(x)\big)^+.$$

Theorem 6.1. *Let $h_1 \in (0,h_2)$ be arbitrary but fixed. Then there exists $\beta_1, \beta_2 \in \mathbb{R}$ such that for almost all $h_0 \in [h_1,h_2]$ there exists $C \in \mathbb{R}$, an increasing sequences $\{N_k\} \subset \mathbb{N}$, a family of increasing sequences $\{i_j^{(k)}\}_{j\geq 1} \subset \mathbb{N}$ parametrised by $k \in \mathbb{N}$, and elements $x_j^k \in P_{N_k}E_{i_j^{(k)}}$ satisfying*

(i) $0 < \tau(x_j^k, x_j^k) \leq C$ *(independently of k,j)*

(ii) $0 < \beta_1 \leq J(h_0, x_j^k) \leq \beta_2$ *(independently of k,j)*

(iii) $2\tau(x_j^k, x)(h_0 - \mathcal{V}(x_j^k)) = \tau(x_j^k, x_j^k)\langle \nabla_{i_j}^{N_k}\mathcal{V}(x_j^k), x\rangle$ *for all $x \in P_{N_k}E_{i_j^{(k)}}$*

(iv) $x_j^k \in \Lambda_{i_j^{(k)}}^{N_k}(h_0)$.

We highlight the fact that β_1 and β_2 depend only on h_1 and h_2, and are independent of $h_0 \in [h_1,h_2]$.

Proof. The idea is to adapt the proof of Theorem 2.1 to find a sequence of points, which are critical points of J with respect to the finite dimensional space $P_{N_k}E_{i_j^{(k)}}$, in such a way that bounds on them are independent of k and j.

By property ($\mathfrak{V}$0) there exists $r > 0$ such that $\mathcal{V}(x) \leq h_1/2$ whenever $\|x\| \leq r$. Hence by property (T2)

$$\inf\{J(h,z) : z \in P_N Z,\ \|P_N z\| = r\} \geq \frac{c_0 h_1 r^2}{2} =: \beta_1$$

for all $h \in [h_1,h_2]$, $N \in \mathbb{N}$. Observe that β_1 is independent of N and $h \in [h_1,h_2]$. Let $\underline{h} \in [h_1,h_2)$ be arbitrary but fixed. Then for $h > \underline{h}$, $N \in \mathbb{N}$,

$$\begin{aligned}
&\sup\{J(h, y+\mu e) : \mu > 0,\ y \in P_N Y,\ y+\mu e \in \Lambda(\underline{h})\}\\
&\quad = \sup\{(\tau(y+\mu e, y+\mu e))^+\big(h - \mathcal{V}(y+\mu e)\big) :\\
&\qquad\qquad\qquad \mu > 0,\ y \in P_N Y,\ y+\mu e \in \Lambda(\underline{h})\}\\
&\quad \leq h\tau_0 \sup\{\mu^2 : \mu > 0,\ y \in P_N Y,\ y+\mu e \in \Lambda(\underline{h})\}, \quad \tau_0 = \tau(e,e)\\
&\quad \leq h\tau_0 \sup\{\mu^2 : \mu > 0,\ y \in Y,\ y+\mu e \in \Lambda(\underline{h})\}\\
&\quad = \tau_0 h M^2.
\end{aligned}$$

Hence for all $h, \underline{h} \in [\underline{h}, h_2]$

$$\sup\{J(h, y + \mu e) : \mu > 0,\ y \in P_N Y,\ y + \mu e \in \Lambda(\underline{h})\} \leq \tau_0 h_2 M^2 =: \beta_2.$$

Observe that β_2 is independent of $N \in \mathbb{N}$ and $h \in [h_1, h_2]$. From (T3),

$$\sup\{J(h, y) : y \in P_N Y \cap \Lambda(\underline{h})\} = 0$$

for all $h \in [h_1, h_2]$ and $N \in \mathbb{N}$. By ($\mathfrak{V}$1) and the fact that $\mathcal{V}(x) = h$ whenever $x \in \partial^N \Lambda^N(h)$, where $\partial^N \Lambda^N(h)$ denotes the boundary of $\Lambda^N(h) := \Lambda(h) \cap P_N X$ with respect to $P_N X$,

$$\sup\{J(h, y + \mu e) : \mu \geq 0,\ y \in P_N Y,\ y + \mu e \in \partial^N \Lambda^N(\underline{h})\} \leq M^2 \tau_0 (h - \underline{h}).$$

Therefore, if $0 < \overline{h} - \underline{h} \leq \frac{c_0 h_1 r^2}{4M^2 \tau_0}$, then

$$\sup\{J(h, y + \mu e) : \mu \geq 0,\ y \in P_N Y,\ y + \mu e \in \partial^N \Lambda^N(\underline{h})\} \leq \beta_1/2$$

for all $h \in [\underline{h}, \overline{h}]$, independently of $h \in [\underline{h}, \overline{h}]$ and $N \in \mathbb{N}$. Let

$$S = \{z \in Z : \|z\| = r\} \quad \text{and} \quad Q(\underline{h}) = \{y + \mu e : y \in Y,\ \mu > 0\} \cap \Lambda(\underline{h}).$$

Note $Q(\underline{h}) \subset \Lambda(h)$ for all $h \in [\underline{h}, \overline{h}]$. For each $i, N \in \mathbb{N}$ put

$$S_i^N = S \cap P_N E_i \quad \text{and} \quad Q_i^N(\underline{h}) = Q(\underline{h}) \cap P_N E_i,$$

and let $\partial_i^N Q(\underline{h})$ denote the boundary of $Q_i^N(\underline{h})$ relative to $\operatorname{span}\{e, P_N E_i \cap Y\}$. Let $\Gamma_i^N(\underline{h})$ denote the set of continuous functions $\gamma : P_N E_i \to P_N E_i$ such that γ coincides with the identity when restricted to $\partial_i^N Q(\underline{h})$. By exactly the same argument as in Lemma 2.3, it follows that S_i^N and $\partial_i^N Q(\underline{h})$ link with respect to $\Gamma_i^N(\underline{h})$. Let

$$c_i^N(h, \underline{h}) = \inf_{\gamma \in \Gamma_i^N(\underline{h})} \max_{x \in Q_i^N(\underline{h})} J(h, \gamma(x))$$

for all $h \in [\underline{h}, \overline{h}]$ and $i, N \in \mathbb{N}$. Since S_i^N and $\partial_i^N Q(\underline{h})$ link it follows that

$$0 < \beta_1 \leq \inf_{x \in S_i^N} J(h, x) \leq c_i^N(h, \underline{h}) \leq \sup_{x \in Q_i^N(\underline{h})} J(h, x) \leq \beta_2, \tag{6.2}$$

for all $h \in [\underline{h}, \overline{h}]$, where β_1 and β_2 are independent of $i, N \in \mathbb{N}$ and $h \in [\underline{h}, \overline{h}]$. By the same argument as Lemma 2.4, the function $h \mapsto c_i^N(h, \underline{h})$ is non-decreasing on $(\underline{h}, \overline{h})$ for all $i, N \in \mathbb{N}$. The monotonicity of $h \mapsto c_i^N(h, \underline{h})$ implies its classical derivative exists and is non-negative almost everywhere in $(\underline{h}, \overline{h})$. Let $c_i^N{}'(h, \underline{h})$ denote $\frac{\partial c_i^N}{\partial h}(h, \underline{h})$ whenever the latter exists. Since c_i^N is non-decreasing,

$$0 \leq \int_{\underline{h}}^{\overline{h}} c_i^N{}'(h, \underline{h})\, \mathrm{d}h \leq c_i^N(\overline{h}, \underline{h}) - c_i^N(\underline{h}, \underline{h}) \leq \beta_2 - \beta_1, \tag{6.3}$$

by (6.2), independently of $i, N \in \mathbb{N}$. Let

$$\underline{c}(h, \underline{h}) = \liminf_{N \to \infty} \liminf_{i \to \infty} c_i^N{}'(h, \underline{h}), \quad h \in [\underline{h}, \overline{h}].$$

Then by two applications of Fatou's lemma and (6.3),

$$0 \leq \int_{\underline{h}}^{\overline{h}} \underline{c}(h, \underline{h})\, \mathrm{d}t \leq \beta_2 - \beta_1.$$

Hence $\underline{c}(h, \underline{h})$ is finite for almost all $h \in [\underline{h}, \overline{h}]$. Choose and fix $h_0 \in (\underline{h}, \overline{h})$ such that $0 \leq \alpha := \underline{c}(h_0, \underline{h}) < \infty$. By the definition of $\underline{c}$ there exists an increasing sequence $\{N_k\}_{k \geq 1} \subset \mathbb{N}$ such that

$$\liminf_{i \to \infty} c_i^{N_k}{}'(h_0, \underline{h}) \leq \alpha + 1/2 \quad \text{for all } k \in \mathbb{N}.$$

Therefore for each $k \in \mathbb{N}$ there exists an increasing sequence $\{i_j^{(k)}\}_{j \geq 1} \subset \mathbb{N}$ such that

$$c_{i_j^{(k)}}^{N_k}{}'(h_0, \underline{h}) \leq \alpha + 1$$

Observe that the real number $\alpha + 1$ depends on $h_0 \in (\underline{h}, \overline{h})$ but not on $k, j \in \mathbb{N}$.

Now that the dependencies of the various real numbers β_1, β_2 and α (and hence C) have been established with respect to E_i, P_N and h, the proof of Theorem 6.1 now proceeds as in the proof of Theorem 2.1 with fixed N_k. □

We now make some additional assumptions. Suppose for almost all $h \in [h_1, h_2]$ and all $N \in \mathbb{N}$

($\mathfrak{H}$1) $\cup_{i \in \mathbb{N}} \{x \in \Lambda_i^N(h) : \nabla_i^N J(h, x) = 0,\, \beta_1 \leq J(h, x) \leq \beta_2,\, 0 < \tau(x, x) \leq C\}$

denoted by $R^N(h)$, is such that $\cup_{N\in\mathbb{N}} R^N$ is a bounded subset of X, whenever $0 < \beta_1 < \beta_2$ and $0 < C$;

($\mathfrak{H}2$) the mapping $F^N : P_N X \to P_N X$ defined by

$$\langle F^N(x), \tilde{x}\rangle = \tau(x,x)\langle \nabla^N \mathcal{V}(x), \tilde{x}\rangle$$

for all $x, \tilde{x} \in P_N X$, is a compact operator;

($\mathfrak{H}3$) if $\{x_i^N\}_{i\geq 1} \subset R^N(h)$ and $\nabla^N J(h, x_i^N) \to 0$ as $i \to \infty$ then $\{x_i^N\}_{i\geq 1}$ has a strongly convergent subsequence (with limit in $P_N X$).

The conditions ($\mathfrak{H}$) differ from (H) (see page 26) by being adapted to the subspaces $P_N X$. This enables the use of Theorem 6.1 with the same argument as in the proof of Theorem 2.7 to obtain the following result.

Theorem 6.2. *Let $\mathcal{V}$ and τ satisfy ($\mathfrak{V}0$) – ($\mathfrak{V}2$) and (T1) – (T4) respectively and suppose ($\mathfrak{H}$) holds. Then there exists $\beta_1, \beta_2 \in \mathbb{R}$ such that for almost all $h_0 \in [h_1, h_2]$ there exists an increasing sequence $\{N_k\} \subset \mathbb{N}$, a real number $C = C(h_0)$ and a bounded sequence $\{x^k\} \subset X$ with $x^k \in P_{N_k} X$ such that*

(i) $0 < \tau(x^k, x^k) \leq C$ *(independently of k)*

(ii) $0 < \beta_1 \leq J(h_0, x^k) \leq \beta_2$ *(independently of k and $h_0 \in [h_1, h_2]$)*

(iii) $2\tau(x^k, x)\big(h_0 - \mathcal{V}(x^k)\big) = \tau(x^k, x^k)\langle \nabla^{N_k}\mathcal{V}(x^k), x\rangle$ *for all $x \in P_{N_k} X$*

(iv) $x^k \in \Lambda^{N_k}(h_0) = \Lambda(h_0) \cap P_{N_k} X$

(v) $\sup_{k\in\mathbb{N}} \|x^k\| < \infty$. □

6.2 Brake periodic orbits

In the following we use Sobolev spaces of Banach-space-valued functions. A summary of such Sobolev spaces can be found in Appendix B. The necessary integration theory is summarised in Appendix A. Let

$$X = \{q : q_{|[0,1]} \in W^{1,2}(0,1; l_2),\ q(2+t) = q(t),\ q(-t) = q(t)\ \forall t \in \mathbb{R}\},$$

which is a real separable Hilbert space (Appendix B) when equipped with the inner product

$$\langle p, q\rangle = \int_0^1 \langle p(t), q(t)\rangle_{l_2} + \langle p'(t), q'(t)\rangle_{l_2} \,\mathrm{d}t.$$

For $N \in \mathbb{N}$ let $P_N : X \to X$ be the projections defined by

$$P_N q = (q_1, q_2, \ldots, q_N, 0, \ldots).$$

Sometimes it is convenient to consider P_N as the analogous projection on l_2. Let

$$E_i = \operatorname{span}\{e_{j,k} : 0 \le k \le i,\ j \in \mathbb{N}\}, \quad i \in \mathbb{N}_0,$$

where, for $j \in \mathbb{N}$ and $k \in \mathbb{N}_0$,

$$e_{j,k}(t) := (0, \ldots, 0, \underbrace{\cos k\pi t}_{j\text{-th coeff.}}, 0, \ldots) \in l_2.$$

It is not necessary to take the closure of the linear span in the definition of E_i. Let $q(t) = (z(t), y(t))$, $q \in X$, where $z(t) \in \mathbb{R}^p$, $p \in \mathbb{N}$ and $y(t) \in l_2$. Note that $\|e_{1,1}\|_X = 1$ and put $(e, 0) = e_{1,1} \in E_1$ where $e(t) \in \mathbb{R}^p$. Put

$$\begin{aligned} Y &= \{(c, y) \in X : c \in \mathbb{R}^p\}, \\ Z &= Y^\perp = \left\{(z, 0) \in X : \int_0^1 z(t)\,\mathrm{d}t = 0\right\}. \end{aligned}$$

In the following, denote the i-th component of $x \in l_2$, with respect to the standard basis on l_2, by x_i or $(x)_i$. Now make the following assumptions on $V : l_2 \to [0, \infty)$:

($\mathfrak{V}'1$) $V \in C^1(l_2, [0, \infty))$ with $V(0) = 0$;

($\mathfrak{V}'2$) $\liminf_{|x|_{l_2} \to \infty} V(x) =: \mathcal{H}_{l_2}(V) > 0$;

($\mathfrak{V}'3$) $\langle z, \partial_z V(x)\rangle \le \widetilde{C}(1 + V(x) + |x|^\gamma)$ for all $x = (z, y) \in l_2$ where $z \in \mathbb{R}^p$, $\widetilde{C} \ge 0$ and $0 < \gamma < 2$;

($\mathfrak{V}'4$) if $x^k \rightharpoonup x$ weakly in l_2 as $k \to \infty$ then $\nabla V(x^k) \rightharpoonup \nabla V(x)$ weakly in l_2 as

$k \to \infty$;

$(\mathfrak{V}'5)$ if $x^k \rightharpoonup x$ weakly in l_2 as $k \to \infty$ then $\liminf_{k\to\infty} V(x^k) \geq V(x)$;

$(\mathfrak{V}'6)$ $\left(\nabla V\right)_i : l_2 \to \mathbb{R}$ is uniformly continuous on bounded sets for all $i \in \mathbb{N}$.

Lemma 6.3. *The condition $(\mathfrak{V}'4)$ implies $\nabla V : l_2 \to l_2$ maps bounded sets into bounded sets.*

Proof. Suppose for a contradiction that $\nabla V : l_2 \to l_2$ does not map bounded sets into bounded sets. Then there exists a bounded sequence $\{x^k\} \subset l_2$ such that

$$\lim_{k\to\infty} |\nabla V(x^k)| = \infty.$$

Since $\{x^k\}$ is bounded it has a weakly convergent subsequence $\{x^{k_j}\}$ with $x^{k_j} \rightharpoonup x$ weakly in l_2 for some $x \in l_2$. By $(\mathfrak{V}'4)$

$$\nabla V(x^{k_j}) \rightharpoonup \nabla V(x)$$

weakly in l_2 as $j \to \infty$. Since weakly convergent sequences are bounded,

$$\sup_{j\in\mathbb{N}} |\nabla V(x^{k_j})| < \infty,$$

which is a contradiction. □

Corollary 6.4. *The potential V maps bounded sets into bounded sets.*

Proof. For all $r > 0$,

$$\sup_{|x|=r} |V(x)| = \sup_{x\in B_r} \left| \int_0^1 \langle \nabla V(tx), x\rangle_{l_2} \, \mathrm{d}t \right| \leq r \sup_{|y|=r} |\nabla V(y)| < \infty.$$

□

Lemma 6.5. *If $q^k \rightharpoonup q$ weakly in X as $k \to \infty$, there exists a subsequence $\{k_j\}$ such that $q^{k_j}(t) \rightharpoonup q(t)$ weakly in l_2, for almost all $t \in [0,1]$, as $j \to \infty$.*

Proof. Firstly, $x^k \rightharpoonup x$ weakly in l_2 as $k \to \infty$ if and only if

$$x^k, x \in l_2, \;\; \sup_k |x^k - x|_{l_2} < \infty \text{ and } x_i^k \to x_i \text{ as } k \to \infty \text{ for every } i \in \mathbb{N}.$$

Now, since $\{q^k\}$ is bounded in X, up to a set of measure zero, $q_i^k : [0,1] \to \mathbb{R}$ are uniformly bounded and equicontinuous for each $i, k \in \mathbb{N}$. Hence, by the Arzela-Ascoli Theorem and a diagonal subsequence argument, there exists a subsequence $\{k_j\}$ such that $q^{k_j}(t) \to q_i(t)$ uniformly as $j \to \infty$ for each $i \in \mathbb{N}$. By Lemma B.3, $\sup_j |q^{k_j}(t) - q(t)|_{l_2} < \infty$ for each $t \in [0,1]$. Hence by the above characterisation of weak convergence in l_2, for all $t \in [0,1]$, $q^{k_j}(t) \rightharpoonup q(t)$ weakly in l_2 as $j \to \infty$. □

Define $\mathcal{V} : X \to [0,\infty)$ and $\tau : X \times X \to \mathbb{R}$ by

$$\mathcal{V}(q) = \int_0^1 V(q(t))\,\mathrm{d}t \quad \text{and} \quad \tau(p,q) = \int_0^1 \langle Sp'(t), q'(t)\rangle_{l_2}\,\mathrm{d}t.$$

Theorem 6.6. *Suppose $V : l_2 \to [0,\infty)$ satisfies $(\mathfrak{V}'1)$ – $(\mathfrak{V}'6)$. Then for every $h \in (0, \mathcal{H}_{l_2}(V))$ there exists $h^* \in (0,h]$ and a non-constant solution u of* (6.1) *with energy h^*.*

Proof. Let $0 < h_1 < h_2 < \mathcal{H}_{l_2}(V)$ be arbitrary but fixed.

Theorem C.1 and $(\mathfrak{V}'1)$ imply $\mathcal{V}$ is well defined and belongs to $C^1(X, [0,\infty))$, verifying $(\mathfrak{V}0)$. Now consider $(\mathfrak{V}1)$. Let $g : [0,\infty) \to [0,\infty)$ be defined by

$$g(\rho) = \inf\{V(x) : |x|_{l_2} \geq \rho\}.$$

Note that $g(0) = 0$ and g is non-decreasing. Suppose $(c,y) + \mu(e,0) \in \Lambda(h_2)$ with $\mu > 0$ and $c \in \mathbb{R}^p$. Then

$$\begin{aligned} h_2 &> \int_0^1 V((c + \mu e(t), y(t)))\,\mathrm{d}t \geq \int_0^1 g(|(c + \mu e(t), y(t))|_{l_2})\,\mathrm{d}t \\ &\geq \int_0^1 g(|c_1 + \mu \cos \pi t|)\,\mathrm{d}t. \end{aligned}$$

By $(\mathfrak{V}'2)$, $\lim_{\rho\to\infty} g(\rho) = \mathcal{H}_{l_2}(V) > h_2$. So, by Lemma 3.8, the set of such μ is bounded. Hence

$$\sup\{\mu > 0 : y + \mu e \in \Lambda(h_2),\ y \in Y\} = M < \infty,$$

for some $M \in \mathbb{R}$.

To show $(\mathfrak{V}2)$, note that for each $i, N \in \mathbb{N}$, $P_N E_i$ is finite dimensional and, by $(\mathfrak{V}'2)$,

$$\liminf_{|P_N x|_{l_2}\to\infty} V(P_N x) \geq \liminf_{|x|_{l_2}\to\infty} V(x) = \mathcal{H}_{l_2}(V).$$

Hence by Lemma 3.8, $\Lambda_i^N(h_2) = \Lambda(h_2) \cap P_N E_i$ is bounded.

To verify condition $(\mathfrak{H}1)$ note $R^N \subset \mathcal{R}$ where

$$\mathcal{R} := \{q \in \Lambda(h) : \nabla_i \mathcal{J}(h,q) = 0, \gamma_1 \leq \mathcal{J}(h,q) \leq \gamma_2, 0 < \tau(q,q) \leq D\},$$

for some $\gamma_2 > \gamma_1 > 0$, $D > 0$. Let $q \in \mathcal{R}$. Then, by the same argument as on page 37,

$$\int_0^1 |q'(t)|^2 \,\mathrm{d}t \leq \frac{D^2}{\gamma_1} \int_0^1 \langle z(t), \partial_z V(q(t))\rangle_{l_2} \,\mathrm{d}t$$

and by $(\mathfrak{V}'3)$,

$$\int_0^1 |q'(t)|^2 \,\mathrm{d}t \leq \frac{D^2 \widetilde{C}}{\gamma_1} \left(1 + h + \left(\int_0^1 |q(t)|_{l_2}^2 \,\mathrm{d}t\right)^{\frac{\gamma}{2}}\right) \tag{6.4}$$

Then $q \in \Lambda(h)$, $(\mathfrak{V}'2)$ and (6.4) imply that the norm of q is bounded independently of N. Hence $(\mathfrak{H}1)$ holds.

The conditions $(\mathfrak{V}'1)$, $(\mathfrak{V}'2)$ and $(\mathfrak{V}'3)$ ensure, by the proof of Theorem 3.9, that the conditions $(\mathfrak{H}2)$ and $(\mathfrak{H}3)$ hold.

Therefore, in the setting of $(\mathfrak{V}'1)$, $(\mathfrak{V}'2)$ and $(\mathfrak{V}'3)$, all the conditions of Theorem 6.2 are satisfied. Hence there exists $\beta_1, \beta_2 \in \mathbb{R}$, dependent of h_1 and h_2, such that for almost all $h_0 \in [h_1, h_2]$ there is an increasing sequence $\{N_k\} \subset \mathbb{N}$, $C \in \mathbb{R}$ and $q^k \in P_{N_k} X$ with $\{q^k\}$ bounded in X,

$$0 < \int_0^1 \langle S q^{k\prime}(t), q^{k\prime}(t)\rangle_{l_2} \,\mathrm{d}t \leq C \text{ independently of } k, \tag{6.5a}$$

$$0 < \beta_1 \leq \int_0^1 \langle S q^{k\prime}(t), q^{k\prime}(t)\rangle_{l_2} \,\mathrm{d}t \int_0^1 h_0 - V(q^k(t)) \,\mathrm{d}t \leq \beta_2 \tag{6.5b}$$

independently of $k \in \mathbb{N}$ and $h_0 \in [h_1, h_2]$,

$$\begin{aligned} 2 \int_0^1 \langle S q^{k\prime}(t), r'(t)\rangle_{l_2} \,\mathrm{d}t \int_0^1 h_0 - V(q^k(t)) \,\mathrm{d}t \\ = \int_0^1 \langle S q^{k\prime}(t), q^{k\prime}(t)\rangle_{l_2} \,\mathrm{d}t \int_0^1 \langle \nabla V(q^k(t)), r(t)\rangle_{l_2} \,\mathrm{d}t \end{aligned} \tag{6.5c}$$

for all $r \in P_{N_k}X$, and

$$0 < \int_0^1 h_0 - V(q^k(t))\,\mathrm{d}t. \tag{6.5d}$$

Since $\{q^k\}$ is bounded in X there is a subsequence, labelled k, and $q \in X$ such that $q^k \rightharpoonup q$ weakly in X and, for each fixed $i \in \mathbb{N}$, $q_i^k \to q_i$ uniformly as $k \to \infty$. Let

$$\lambda_k^2 = \int_0^1 \langle S{q^k}'(t), {q^k}'(t)\rangle_{l_2}\,\mathrm{d}t \left(2\int_0^1 h_0 - V(q^k(t))\,\mathrm{d}t\right)^{-1}. \tag{6.6}$$

By (6.5a), (6.5b), (6.5d) and since the potential V is non-negative, $\lambda_k \in [\frac{\beta_1}{h_0}, \frac{C^2}{2\beta_1}]$. So there exists a subsequence, labelled k, such that $\{\lambda_k\}$ converges to some $\lambda \in [\frac{\beta_1}{h_0}, \frac{C^2}{2\beta_1}]$. By (6.5c)

$$\int_0^1 \langle S{q^k}'(t), r'(t)\rangle_{l_2}\,\mathrm{d}t = \lambda_k^2 \int_0^1 \langle P_{N_k}\nabla V(q^k(t)), r(t)\rangle_{l_2}\,\mathrm{d}t$$

for all $r \in X$. Hence for each $i \in \mathbb{N}$,

$$\int_0^1 s_i {q_i^k}'(t)\phi'(t)\,\mathrm{d}t = \lambda_k^2 \int_0^1 \left(\nabla V(q^k(t))\right)_i \phi(t)\,\mathrm{d}t \tag{6.7}$$

for all $\phi \in C_0^\infty(0,1;\mathbb{R})$ and all k with $N_k \geq i$.

Since $q^k \rightharpoonup q$ weakly in X, ${q^k}' \rightharpoonup q'$ weakly in $L^2(0,1;l_2)$ and so ${q_i^k}' \rightharpoonup q_i'$ weakly in $L^2(0,1;\mathbb{R})$. Let $\{e_i : i \in \mathbb{N}\}$ be a basis for l_2. Then for each fixed $i \in \mathbb{N}$ and $\phi \in C_0^\infty(0,1;\mathbb{R})$,

$$\int_0^1 s_i {q_i^k}'(t)\phi'(t)\,\mathrm{d}t \to \int_0^1 s_i q_i'(t)\phi'(t)\,\mathrm{d}t \tag{6.8}$$

as $k \to \infty$. Now, since $\{q^k\}$ is bounded in X, $\{q^k(t) : k \in \mathbb{N},\ t \in [0,1]\}$ is, by Lemma B.3, bounded in l_2. So, by ($\mathfrak{V}'4$) and Lemma 6.3, $\{\nabla V(q^k(t)) : k \in \mathbb{N},\ t \in [0,1]\}$ is bounded in l_2. Since $q^k \rightharpoonup q$ weakly in X, by Lemma 6.5, there is a subsequence, labelled k, such that $q^k(t) \rightharpoonup q(t)$ weakly in l_2 for all $t \in [0,1]$. By ($\mathfrak{V}'4$),

$$\nabla V(q^k(t)) \rightharpoonup \nabla V(q(t))$$

weakly in l_2 for all $t \in [0,1]$. Hence for each $i \in \mathbb{N}$,

$$\big(\nabla V(q^k(t))\big)_i \to \big(\nabla V(q(t))\big)_i$$

for all $t \in [0,1]$ as $j \to \infty$. So by Lemma 6.3 and the Dominated Convergence Theorem, for all $i \in \mathbb{N}$ and all $\phi \in C_0^\infty(0,1;\mathbb{R})$,

$$\int_0^1 \big(\nabla V(q^k(t))\big)_i \phi(t)\,\mathrm{d}t \to \int_0^1 \big(\nabla V(q(t))\big)_i \phi(t)\,\mathrm{d}t \tag{6.9}$$

as $j \to \infty$. Hence by (6.7), (6.8) and (6.9),

$$\int_0^1 \langle Sq'(t), \phi'(t)e_i\rangle\,\mathrm{d}t = \lambda^2 \int_0^1 \langle \nabla V(q(t)), \phi(t)e_i\rangle\,\mathrm{d}t,$$

for all $\phi \in C_0^\infty(0,1;\mathbb{R})$ and all $i \in \mathbb{N}$. So by Lemma B.2, $q' \in X$ and

$$Sq'' + \lambda^2 \nabla V(q) \equiv 0. \tag{6.10}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\frac{1}{2}\langle Sq'(t), q'(t)\rangle_{l_2} + \lambda^2 V(q(t))\} \equiv 0.$$

Hence

$$\frac{1}{2}\langle Sq', q'\rangle_{l_2} + \lambda^2 V(q) \equiv \lambda^2 h^*, \quad \text{for some } h^* \in \mathbb{R}. \tag{6.11}$$

By the definition of X, $q'(0) = 0$, so $h^* = V(q(0))$. By (6.7) and Lemma B.2, q^k is twice continuously differentiable and

$$Sq^{k\,\prime\prime} + \lambda_k^2 P_{N_k} \nabla V(q^k) \equiv 0. \tag{6.12}$$

Hence

$$\frac{1}{2}\langle Sq^{k\,\prime}, q^{k\,\prime}\rangle_{l_2} + \lambda_k^2 V(q^k) \equiv h_0 \lambda_k^2, \tag{6.13}$$

where the constant of integration is given by another integration and (6.6). Therefore, by (6.13) and since $q^{k\,\prime}(0) = 0$, $V(q^k(0)) = h_0$. Since $q^k(0) \rightharpoonup q(0)$ weakly in l_2 as $k \to \infty$, by ($\mathfrak{V}'5$),

$$h_0 = \liminf_{k\to\infty} V(q^k(0)) \geq V(q(0)) = h^*, \tag{6.14}$$

and consequently, by (6.11),

$$\frac{1}{2}\langle Sq', q'\rangle_{l_2} + \lambda^2 V(q) \equiv \lambda^2 h^* \leq \lambda^2 h_0. \tag{6.15}$$

Next we show that q is non constant. By (6.5a) and (6.5d)

$$0 < \beta_1 \leq h_0 \int_0^1 \langle Sq^{k\,\prime}(t), q^{k\,\prime}(t)\rangle_{l_2} \mathrm{d}t. \tag{6.16}$$

If, for each $i \in \mathbb{N}$, $q_i^{k\,\prime} \to q_i'$ uniformly as $k \to \infty$ then

$$\begin{aligned} 0 < \frac{\beta_1}{h_0} &\leq \limsup_{k\to\infty} \int_0^1 \langle Sq^{k\,\prime}(t), q^{k\,\prime}(t)\rangle_{l_2} \mathrm{d}t, \\ &= \limsup_{k\to\infty} \int_0^1 \sum_{i=1}^{p} |q_i^{k\,\prime}(t)|^2 \,\mathrm{d}t - \liminf_{k\to\infty} \int_0^1 \sum_{i=p+1}^{\infty} |q_i^{k\,\prime}(t)|^2 \,\mathrm{d}t \end{aligned}$$

and by bounded convergence,

$$= \int_0^1 \sum_{i=1}^{p} \limsup_{k\to\infty} |q_i{}'(t)|^2 \,\mathrm{d}t - \liminf_{k\to\infty} \int_0^1 \sum_{i=p+1}^{\infty} |q_i^{k\,\prime}(t)|^2 \,\mathrm{d}t,$$

then, by two applications of Fatou's Lemma,

$$\begin{aligned} &\leq \int_0^1 \sum_{i=1}^{p} |q_i{}'(t)|^2 \,\mathrm{d}t - \int_0^1 \sum_{i=p+1}^{\infty} \liminf_{k\to\infty} |q_i^{k\,\prime}(t)|^2 \,\mathrm{d}t, \\ &= \int_0^1 \langle Sq'(t), q'(t)\rangle_{l_2} \mathrm{d}t. \end{aligned} \tag{6.17}$$

Now we prove there is a subsequence, labelled k, such that for each $i \in \mathbb{N}$

$$q_i^{k\,\prime} \to q_i' \text{ uniformly as } k \to \infty.$$

By (6.12), q_i^k is twice continuously differentiable and satisfies

$$q_i^{k\,\prime\prime} + \lambda_k^2 \big(\nabla V(q^k)\big)_i = 0, \tag{6.18a}$$

$$q_i^{k\,\prime}(0) = 0, \tag{6.18b}$$

whenever $k \geq i$. Since $q^k \rightharpoonup q$ pointwise weakly in l_2 as $k \to \infty$, and since ∇V is weakly continuous, for all $t \in [0,1]$

$$\big(\nabla V(q^k(t))\big)_i \to \big(\nabla V(q(t))\big)_i \quad as\ k \to \infty.$$

However $\{q^k\}$ is uniformly bounded and equicontinuous on $[0,1]$. Since, by ($\mathfrak{V}'6$), $\big(\nabla V\big)_i$ is uniformly continuous on bounded sets in l_2, the composition $\{\big(\nabla V(q^k)\big)_i\}_{k\in\mathbb{N}}$ is uniformly bounded and equicontinuous from $[0,1]$ to $\mathbb{R}$. The Arzela-Ascoli Theorem and a diagonal subsequence argument imply that there exists a subsequence, labelled k, such that

$$\big(\nabla V(q^k(t))\big)_i \to \big(\nabla V(q(t))\big)_i \ \text{ uniformly as } k \to \infty$$

for each $i \in \mathbb{N}$. Therefore (6.10) and (6.18a) imply

$$q_i^{k\,\prime\prime} \to -\lambda^2\big(\nabla V(q)\big)_i \ \text{ uniformly as } k \to \infty.$$

Hence by (6.18b) and [Apo57, Theorem 13–13, p402]

$$q_i^{k\,\prime} \to q_i{}' \ \text{ uniformly as } k \to \infty.$$

Therefore (6.17) holds completing the proof that q is non-constant.

To show that q yields a solution of (6.1), let $u(t) = q(t/\lambda)$ for all $t \in \mathbb{R}$. Then $u \in C^2(l_2, [0,\infty))$ and, by (6.10) and (6.15),

$$\begin{gathered} Su''(t) + \nabla V(u(t)) = 0, \\ \frac{1}{2}\langle Su'(t), u'(t)\rangle_{l_2} + V(u(t)) = h^*, \text{and} \\ u'(0) = u'(\lambda) = 0. \end{gathered}$$

By (6.17), $\frac{1}{2}\langle Su'(t), u'(t)\rangle_{l_2} > 0$ for some $t \in \mathbb{R}$. Therefore $h^* > 0$ as the potential V is non-negative. Finally, the proof follows since h_1 and h_2 were chosen arbitrarily in $(0, \mathcal{H}_{l_2}(V))$. ☐

Corollary 6.7. *If in addition to ($\mathfrak{V}'1$)–($\mathfrak{V}'6$),*

$$\big(\nabla V(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots)\big)_j = 0 \quad implies \quad P_p x = 0,$$

then $u_j \not\equiv 0$.

Proof. This follows from (6.10) since the solutions u arising from Theorem 6.6 satisfy $\int_0^{t_0} \langle Su'(t), u'(t)\rangle_{l_2}\,dt > 0$ by definition of u and (6.17). $\square$

The fact that $(\mathfrak{V}'1)$–$(\mathfrak{V}'6)$ and the condition of Corollary 6.7 are compatible conditions is proved by the following example.

Example 6.8. *Let $g \in C^1(l_2, [0,1))$ have bounded gradient and suppose $g(P_p x) = 0$ if and only if $P_p x = 0$. Let $f \in C^1(\mathbb{R}, [0,1))$ be such that $1 > f'(s) > 0$ for all $s \in \mathbb{R}$ and let $\alpha \in l_2$ with $\alpha_i > 0$ whenever $i > p$. Now define $V : l_2 \to [0,\infty)$ by*

$$V(x) = g(P_p x) f(\langle \alpha, x\rangle_{l_2}) + |x|_{l_2}^2 \quad \textit{for all } x \in l_2.$$

Then $V(0) = 0$ and $V \in C^1(l_2, [0,\infty))$ with

$$\nabla V(x) = f(\langle \alpha, x\rangle_{l_2}) P_p \nabla g(P_p x) + g(P_p x) f'(\langle \alpha, x\rangle_{l_2})\alpha + 2x.$$

It is clear that V satisfies the conditions $(\mathfrak{V}'1)$–$(\mathfrak{V}'6)$. To verify the condition of Corollary 6.7, put $x^{(j)} = (x_1, x_2, \ldots, x_{j-1}, 0, x_{j+1}, \ldots)$. Then for all $j > p$,

$$\begin{aligned}\left(\nabla V(x^{(j)})\right)_j &= g(P_p x) f'(\langle \alpha, x^{(j)}\rangle_{l_2})\alpha_j \\ &= 0 \quad \textit{if and only if} \quad P_p x = 0.\end{aligned}$$

Remark 6.9. *If in addition to $(\mathfrak{V}'1)$–$(\mathfrak{V}'6)$, V is weakly continuous (in the sense that $V(x^k) \to V(x)$ whenever $x^k \rightharpoonup x$ weakly as $k \to \infty$), then (6.14) holds with equality. This would imply that there is a non-trivial solution of (6.1) for all $h \in (0, \mathcal{H}_{l_2}(V))$. However $(\mathfrak{V}'1)$ and $(\mathfrak{V}'2)$ are incompatible with V being weakly continuous. To show this suppose both hold. By $(\mathfrak{V}'2)$ there exists $r > 0$ and $\varepsilon > 0$ such that*

$$\inf_{|x|_{l_2} = r} V(x) \geq \varepsilon.$$

Let $x^n = re_n$, where $\{e_n\}$ is an orthonormal basis for l_2, then $|x^n|_{l_2} = r$ and $x^n \rightharpoonup 0$ weakly in l_2 as $n \to \infty$. However, by $(\mathfrak{V}'1)$ and the weak continuity of V,

$$0 = V(0) = \lim_{n\to\infty} V(x^n) \geq \liminf_{n\to\infty} V(x^n) \geq \varepsilon > 0,$$

which is a contradiction. A consequence of this is that ∇V is not permitted to be compact as this implies V is weakly continuous by [Vai64, Theorem 8.2, p.76].

Chapter 7

Even Potentials in Infinite Dimensions

This chapter brings together the theory of even potentials defined on finite dimensional spaces in Chapter 4 and the theory of potentials defined on l_2 in Chapter 6. An even potential defined on l_2 is considered. We seek odd periodic $u : \mathbb{R} \to l_2$ satisfying

$$Su''(t) + \nabla V(u(t)) = 0, \tag{7.1a}$$

$$\frac{1}{2}\langle Su'(t), u'(t)\rangle_{l_2} + V(u(t)) = h, \tag{7.1b}$$

$$u'(t_0) = u(0) = 0, \tag{7.1c}$$

for all $t \in \mathbb{R}$, where $V \in C^1(l_2, \mathbb{R})$ is even and $t_0 > 0$. The potential is normalised so that $V(0) = 0$. As in Chapter 6 the operator $S : l_2 \to l_2$ is given by $(Sx)_i = s_i x_i$ for all $i \in \mathbb{N}$ with

$$s_i = \begin{cases} +1 & \text{if } 1 \leq i \leq p, \\ -1 & \text{if } p \leq i, \end{cases}$$

for some fixed $p \in \mathbb{N}$.

By (7.1b) and (7.1c) there are no constant solutions of (7.1) unless $h^* = 0$; if $h^* = 0$ the only constant solution is the zero solution. We show, when the potential is even, there exists a solution of (7.1) for almost all $h \in (0, \mathcal{H}_{W,l_2}(V))$ where

$$\mathcal{H}_{W,l_2}(V) = \liminf_{|P_W x|_{l_2} \to \infty} V(x) > 0,$$

W is a subspace of l_2 with co-dimension $p-1$ containing the negative eigenspace of S, and P_W is the orthogonal projection of l_2 onto W.

7.1 Abstract theory

Let X be a separable Hilbert space and $X \neq Y \subset X$ be a closed subspace so that $X = Y \oplus Z$ where $Z = Y^{\perp}$. Let E_i be subspaces of X, $i \in \mathbb{N}$. We do not assume that the E_i are finite dimensional. Let $P_N : X \to X$ be such that

(i) $P_N : X \to X$ is a projection for all $N \in \mathbb{N}$,

(ii) $P_N E_i = (P_N E_i \cap Y) \oplus (P_N E_i \cap Z)$ for all $i, N \in \mathbb{N}$,

(iii) $E_i \subset E_{i+1}$ for all $i \in \mathbb{N}$,

(iv) $\bigcup_{i \in \mathbb{N}} P_N E_i$ is dense in $P_N X$, and

(v) $P_N E_i$ is finite dimensional for all $i, N \in \mathbb{N}$.

By (i) it follows that $P_N X$ is closed in X. Let $\nabla u(x)$ denote the gradient at $x \in X$, with respect to the inner product $\langle \cdot, \cdot \rangle$ in X, of a C^1 functional $u : X \to \mathbb{R}$. If $x \in P_N E_i$ let $\nabla_i^N u(x) \in P_N E_i$ denote the gradient of its restriction to $P_N E_i$, and if $x \in P_N X$ let $\nabla^N u(x) \in P_N X$ denote the gradient of its restriction to $P_N X$ with respect to the same inner product.

Let $0 < h_2 < \infty$, $e \in P_N E_1 \cap Z$ for all N sufficiently large, $\|e\| = 1$, $\mathcal{V} : X \to [0, \infty)$ be a functional and for any $h \in [0, \infty)$ denote

$$\Lambda(h) = \{x \in X : \mathcal{V}(x) < h\} \quad \text{and} \quad \Lambda_i^N(h) = \Lambda(h_2) \cap P_N E_i.$$

Let $P_{i,Y}$ denote the orthogonal projection of E_i onto $Y \cap E_i$; the subscript Y has been added to avoid confusion with the projection P_N. Suppose $\mathcal{V} \in C^1(X, \mathbb{R})$

satisfies the following properties:

($\mathfrak{V}0^*$) $\mathcal{V}(y+\mu e) \geq 0 = \mathcal{V}(0)$ whenever $\mu \geq 0$ and $y \in Y$;

($\mathfrak{V}1$) $\sup\{\mu > 0 : y+\mu e \in \Lambda(h_2),\, y \in Y\} = M < \infty$;

($\mathfrak{V}2^*$) $\{y+\mu e : y \in Y,\, \mu > 0\} \cap \Lambda_i^N(h_2)$ is bounded in X for all $i, N \in \mathbb{N}$;

($\mathfrak{V}3^*$) $P_{i,Y}\Lambda_i^N(h_2)$ is bounded for all $i, N \in \mathbb{N}$.

Suppose $\tau : X \times X \to \mathbb{R}$ is a continuous, symmetric, bilinear functional with

(T1) $\tau(y,z) = 0$ for all $(y,z) \in Y \times Z$;

(T2) $\tau(z,z) \geq c_0\|z\|^2 > 0$ for all $z \in Z \setminus \{0\}$;

(T3*) $0 \leq -\tau(y,y) \leq c_0\|y\|^2$ for all $y \in Y$;

(T4) $\tau(x,\tilde{x}) = 0$ for all $(x,\tilde{x}) \in P_N E_i \times P_N(E_i^\perp)$ for all $i, N \in \mathbb{N}$.

For all $h \in (0,h_2)$ and $x \in X$, define

$$\mathcal{J}(h,x) = \tau(x,x)\big(h - \mathcal{V}(x)\big) \quad \text{and} \quad J(h,x) = (\tau(x,x))^+\big(h - \mathcal{V}(x)\big)^+.$$

The following theorem is a straightforward adaptation of Section 4.1 to the infinite dimensional case; the argument is the same as that of Section 6.1 where the existence theory of Section 2.1 is extended to infinite dimensions.

Theorem 7.1. *Let $\mathcal{V}$ and τ satisfy ($\mathfrak{V}0^*$), ($\mathfrak{V}1$), ($\mathfrak{V}2^*$), ($\mathfrak{V}3^*$), (T1), (T2), (T3*) and (T4), and suppose (see page 78) ($\mathfrak{H}$) holds. Fix $h_1 \in (0,h_2]$. Then there exists $\beta_1, \beta_2 \in \mathbb{R}$ such that for almost all $h_0 \in [h_1,h_2]$ there exists an increasing sequence $\{N_k\} \subset \mathbb{N}$, a real number $C = C(h_0)$ and a bounded sequence $\{x^k\} \subset X$ with $x^k \in P_{N_k}X$ satisfying*

(i) $0 < \tau(x^k,x^k) \leq C$ *(independently of k)*

(ii) $0 < \beta_1 \leq J(h_0,x^k) \leq \beta_2$ *(independently of k and $h_0 \in [h_1,h_2]$)*

(iii) $2\tau(x^k,x)\big(h_0 - \mathcal{V}(x^k)\big) = \tau(x^k,x^k)\langle \nabla^{N_k}\mathcal{V}(x^k), x\rangle$ *for all $x \in P_{N_k}X$*

(iv) $x^k \in \Lambda^{N_k}(h_0) = \Lambda(h_0) \cap P_{N_k}X$

(v) $\sup_{k\in\mathbb{N}} \|x^k\| < \infty$. □

7.2 Brake periodic orbits

Let X be the real separable Hilbert space defined by

$$X = \{q : q_{|[0,1]} \in W^{1,2}(0,1,l_2),\ q(1-t) = q(1+t),\ q(-t) = -q(t)\,\forall t \in \mathbb{R}\}$$

with inner product

$$\langle p, q\rangle = \int_0^1 \langle p'(t), q'(t)\rangle_{l_2}\, dt.$$

For $N \in \mathbb{N}$ let $P_N : X \to X$ be the projections defined by

$$P_N q = (q_1, q_2, \ldots, q_N, 0, \ldots).$$

Let

$$E_i = \operatorname{span}\{e_{j,k} : 0 \le k \le i,\ j \in \mathbb{N}\}, \quad i \in \mathbb{N}_0,$$

where, for $j \in \mathbb{N}$ and $k \in \mathbb{N}_0$,

$$e_{j,k}(t) := (0, \ldots, 0, \underbrace{\sin k\pi t}_{j\text{-th coeff.}}, 0, \ldots) \in l_2.$$

For each $q \in X$ write $q(t) = (z(t), y(t))$ where $z(t) \in \mathbb{R}^p$, $p \in \mathbb{N}$ and $y(t) \in l_2$. Define

$$Y = \{(0, y) \in X\} \ \text{ and } \ Z = Y^{\perp} = \{(z, 0) \in X\}.$$

Let $\underline{e} \in \mathbb{R}^p$ satisfy $|\underline{e}| = 2$. Put $(e,0)(t) = (\underline{e}, 0)\sin \pi t$ then $(e,0) \in P_N E_1 \cap Z$ for all $N \ge p$. Let

$$W = \{(\mu\underline{e}, c) : \mu \in \mathbb{R},\ c \in l_2\} \subset l_2$$

and let P_W be the orthogonal projection of l_2 onto W. Let $V \in C^1(l_2, \mathbb{R})$ be even, define

$$\mathcal{H}_{W,l_2}(V) = \liminf_{|P_W x|_{l_2} \to \infty} V(x)$$

and suppose

($\mathfrak{V}'1^*$) $V(0) = 0 \le V(x)$ for all $x \in W$;

($\mathfrak{V}'2^*$) $0 < \mathcal{H}_{W,l_2}(V)$;

($\mathfrak{V}'3$) $\langle z, \partial_z V(x)\rangle \le \widetilde{C}(1 + V(x) + |x|^\gamma)$ for all $x = (z, y) \in l_2$ where $z \in \mathbb{R}^p$, $\widetilde{C} \ge 0$ and $0 < \gamma < 2$;

($\mathfrak{V}'4$) if $x^k \rightharpoonup x$ weakly in l_2 as $k \to \infty$ then $\nabla V(x^k) \rightharpoonup \nabla V(x)$ weakly in l_2 as $k \to \infty$;

($\mathfrak{V}'5$) if $x^k \rightharpoonup x$ weakly in l_2 as $k \to \infty$ then $\liminf_{k\to\infty} V(x^k) \ge V(x)$;

($\mathfrak{V}'6$) for each $i \in \mathbb{N}$, $\big(\nabla V\big)_i : l_2 \to \mathbb{R}$ is uniformly continuous on bounded sets.

Define $\mathcal{V} : X \to [0, \infty)$ and $\tau : X \times X \to \mathbb{R}$ by

$$\mathcal{V}(q) = \int_0^1 V(q(t))\,\mathrm{d}t \quad \text{and} \quad \tau(p, q) = \int_0^1 \langle Sp'(t), q'(t)\rangle_{l_2}\,\mathrm{d}t.$$

Then τ clearly satisfies the conditions (T1), (T2), (T3*) and (T4) with $c_0 = 1$.

Theorem 7.2. *Suppose $V \in C^1(l_2, \mathbb{R})$ is even and satisfies ($\mathfrak{V}'1^*$) - ($\mathfrak{V}'2^*$), ($\mathfrak{V}'3$) - ($\mathfrak{V}'6$). Then there exists a solution u of (7.1) for almost all $h \in (0, \mathcal{H}_{W,l_2}(V))$.*

Proof. Firstly we show all the conditions of Theorem 7.1 are satisfied. Let $0 < h_1 < h_2 < \mathcal{H}_{W,l_2}(V)$ be arbitrary but fixed. Condition ($\mathfrak{V}'1^*$) clearly implies ($\mathfrak{V}0^*$) since $\mu(e(t), 0) + (0, y(t)) \in W$ for all $t \in \mathbb{R}$ and $(0, y) \in Y$.

Let $h \in (0, \mathcal{H}_{W,l_2}(V))$ be arbitrary but fixed. Define $g_W : [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ by

$$g_W(\rho) = \inf_{|P_w x|_{l_2} \ge \rho} V(x).$$

Then g_W is increasing and $\lim_{\rho\to\infty} g_W(\rho) = \mathcal{H}_{W,l_2}(V)$. Suppose $(\mu e, y) \in \Lambda(h)$ where $\mu > 0$ and $(0, y) \in Y$. Then by definition of $\Lambda(h)$

$$\begin{aligned} h &> \int_0^1 V(\mu e, y)\,\mathrm{d}t \ge \int_0^1 g_W(|(\mu e, y)|_{l_2})\,\mathrm{d}t \\ &\ge \int_0^1 g_W(2|\mu| \sin \pi t)\,\mathrm{d}t. \end{aligned}$$

Hence $\sup\{\mu > 0 : (\mu e, y) \in \Lambda(h),\ (0,y) \in Y\} < \infty$ and so ($\mathfrak{V}1$) holds.

Suppose there exists $\{q^k\} \subset \{\mu(e,0) + (0,y) \in \Lambda_i^N(h) : (0,y) \in Y\}$, for fixed $i, N \in \mathbb{N}$, such that $\|q^k\| \to \infty$ as $k \to \infty$. Since $\{q^k\}$ is a sequence of analytic functions in a finite dimensional space the conditions of Lemma 3.8 are satisfied and so there exists an increasing sequence $\{k_j\} \subset \mathbb{N}$ and $U \subset [0,1]$ with $\operatorname{meas} U > h/\mathcal{H}_{W,l_2}(V)$ such that $|q^{k_j}|_{l_2} \to \infty$ uniformly for $t \in U$. Then, since $q^{k_j} : \mathbb{R} \to W$,

$$h > \int_0^1 V(q^{k_j}(t))\, \mathrm{d}t \geq \int_U g_W(|q^{k_j}(t)|_{l_2})\, \mathrm{d}t \to \mathcal{H}_{W,l_2}(V) \operatorname{meas} U$$

for as $j \to \infty$. This contradicts the fact that $\operatorname{meas} U > h/\mathcal{H}_{W,l_2}(V)$ and so completes the proof of ($\mathfrak{V}2^*$).

Suppose, for fixed $i, N \in \mathbb{N}$, there exists $\{(z^k, y^k)\} \subset \{(z,y) \in \Lambda_i^N(h) : (z,y) \in Z \times Y\}$ such that $\|P_{i,Y}(z^k,y^k)\| = \|(0,y^k)\| \to \infty$ as $k \to \infty$. Then there exists an increasing sequence $\{k_j\} \subset \mathbb{N}$ and $U \subset [0,1]$ with $\operatorname{meas} U > h/\mathcal{H}_{W,l_2}(V)$ such that $|(0,y^{k_j})|_{l_2} \to \infty$ uniformly for $t \in U$. Then

$$h > \int_0^1 V((z^{k_j}, y^{k_j}(t)))\, \mathrm{d}t \geq \int_U g_W(|(0,y^{k_j}(t))|_{l_2})\, \mathrm{d}t \to \mathcal{H}_{W,l_2}(V) \operatorname{meas} U$$

for as $j \to \infty$. This contradicts the fact that $\operatorname{meas} U > h/\mathcal{H}_{W,l_2}(V)$ and so completes the proof of ($\mathfrak{V}3^*$).

To verify ($\mathfrak{H}1$), let $\gamma_2 > \gamma_1 > 0$, $D > 0$ be arbitrary and fixed. Recall the set $\mathcal{R}^N$ is given by

$$\mathcal{R}^N = \cup_{i \in \mathbb{N}} \{q \in \Lambda_i^N(h) : \nabla_i^N \mathcal{J}(h,q) = 0, \gamma_1 \leq \mathcal{J}(h,q) \leq \gamma_2, 0 < \tau(q,q) \leq D\}.$$

Let $q \in R^N$, then ($\mathfrak{V}'3$) implies

$$\int_0^1 |q'(t)|^2\, \mathrm{d}t < \frac{D^2 \widetilde{C}}{\gamma_1} \left(1 + h + \left(\int_0^1 |q'(t)|^2\, \mathrm{d}t\right)^{\frac{\gamma}{2}}\right),$$

and so R^N is bounded in X independently of N. Hypotheses ($\mathfrak{H}2$) and ($\mathfrak{H}3$) follow as in the proof of Theorem 3.9.

We have shown that all the conditions of Theorem 7.1 are satisfied. Hence there exists $\beta_1, \beta_2 \in \mathbb{R}$, dependent of h_1 and h_2, such that for almost all $h_0 \in$

$[h_1, h_2]$ there is an increasing sequence $\{N_k\} \subset \mathbb{N}$, $C \in \mathbb{R}$ and $q^k \in P_{N_k}X$ with $\{q^k\}$ bounded in X,

$$0 < \int_0^1 \langle Sq^{k\prime}(t), q^{k\prime}(t)\rangle_{l_2}\,\mathrm{d}t \le C \text{ independently of } k, \tag{7.2a}$$

$$0 < \beta_1 \le \int_0^1 \langle Sq^{k\prime}(t), q^{k\prime}(t)\rangle_{l_2}\,\mathrm{d}t \int_0^1 h_0 - V(q^k(t))\,\mathrm{d}t \le \beta_2 \tag{7.2b}$$

independently of $k \in \mathbb{N}$ and $h_0 \in [h_1, h_2]$,

$$\begin{aligned} 2\int_0^1 \langle Sq^{k\prime}(t), r'(t)\rangle_{l_2}\,\mathrm{d}t \int_0^1 h_0 - V(q^k(t))\,\mathrm{d}t \\ = \int_0^1 \langle Sq^{k\prime}(t), q^{k\prime}(t)\rangle_{l_2}\,\mathrm{d}t \int_0^1 \langle \nabla V(q^k(t)), r(t)\rangle_{l_2}\,\mathrm{d}t \end{aligned} \tag{7.2c}$$

for all $r \in P_{N_k}X$, and

$$0 < \int_0^1 h_0 - V(q^k(t))\,\mathrm{d}t.$$

Since $\{q^k\}$ is bounded in X there exists a subsequence, labelled k, such that $q^k \rightharpoonup q$ weakly in X for some $q \in X$ with $q_i^k \to q_i$ uniformly as $k \to \infty$. Let

$$\lambda_k^2 = \int_0^1 \langle Sq^{k\prime}(t), q^{k\prime}(t)\rangle_{l_2}\,\mathrm{d}t \left(2\int_0^1 h_0 - V(q^k(t))\,\mathrm{d}t\right)^{-1}.$$

The fact that λ_k is uniformly bounded above follows immediately from (7.2a) and (7.2b). Since $\{q^k\}$ is bounded in X, Lemma B.3 implies $|q^k(t)|_{l_2}$ is uniformly bounded in t and k. Therefore $V(q^k(t))$ is uniformly bounded in t and k since V maps bounded sets into bounded sets by Corollary 6.4. Hence

$$0 < \int_0^1 h_0 - V(q^k(t))\,\mathrm{d}t \le \sup_{k\in\mathbb{N}} \int_0^1 h_0 - V(q^k(t))\,\mathrm{d}t =: M < \infty$$

and consequently, by (7.2a), $\lambda_k^2 \ge M^{-2}\beta_1/2$ for all $k \in \mathbb{N}$. So choose a subsequence, labelled k, such that $\lambda_k \to \lambda$ for some $\lambda > 0$. By the same argument as in the proof of Theorem 6.6, $q' \in X$, ${q_i^k}' \to {q_i}'$ uniformly as $k \to \infty$,

$$Sq'' + \lambda^2 \nabla V(q) \equiv 0 \quad \text{and} \quad \int_0^1 \langle Sq'(t), q'(t)\rangle_{l_2}\, \mathrm{d}t > 0.$$

Therefore

$$\frac{1}{2}\langle Sq', q'\rangle_{l_2} + \lambda^2 V(q) \equiv \lambda^2 h^*$$

for some $h^* \in \mathbb{R}$. By (7.2c), q^k satisfies

$$\frac{1}{2}\langle S{q^k}', {q^k}'\rangle_{l_2} + \lambda_k^2 V(q^k) \equiv h_0 \lambda_k^2.$$

By ($\mathfrak{V}'5$) and since $q'(1) = 0 = {q^k}'(1)$,

$$h^* = V(q(1)) \leq \liminf_{k\to\infty} V(q^k(1)) = h.$$

Finally, since $q(0) = 0 = q^k(0)$,

$$h = \liminf_{k\to\infty} \frac{1}{2\lambda_k^2}\langle S{q^k}'(0), {q^k}'(0)\rangle_{l_2} \leq \frac{1}{2\lambda^2}\langle Sq'(0), q'(0)\rangle_{l_2} = h^*.$$

Hence $h^* = h$ and the proof of the theorem is complete. □

The following example shows that the conditions of Theorem 7.2 can be satisfied and the resulting solution has infinitely many non-zero components.

Example 7.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded, smooth, strictly increasing function such that $f(r) = 0$ only if $r = 0$. Let $\{a^i\}$, $\{b^i\} \subset l_2$ satisfy*

$$\sum_{i\in\mathbb{N}} \|a^i\|_{l_2}\|b^i\|_{l_2} < \infty.$$

For $x \in l_2$ define

$$V(x) = \frac{1}{2}\|P_W x\|^2 + Af\big(\sum_{i\in\mathbb{N}} \langle a^i, x\rangle_{l_2}\langle b^i, x\rangle_{l_2}\big).$$

Then the potential V is even, lies in $C^1(l_2, \mathbb{R})$, is weakly lower semi-continuous and its gradient is weakly continuous. Moreover, for $A \neq 0$ sufficiently small, the *potential V is positive on $P_W l_2$. The gradient of the potential V is given by*

$$\nabla V(x) = P_W x + Af'\big(\sum_{i\in\mathbb{N}}\langle a^i, x\rangle_{l_2}\langle b^i, x\rangle_{l_2}\big)\sum_{i\in\mathbb{N}}\big(a^i\langle b^i, x\rangle_{l_2} + b^i\langle a^i, x\rangle_{l_2}\big) \tag{7.3}$$

for all $x \in l_2$. We make the following assumptions on $\{a^i\}$ and $\{b^i\}$.

(a) If $\langle a^i, x\rangle_{l_2} = 0$ for all $i \in \mathbb{N}$ then $P_p x = 0$;

(b) if $\{\alpha_i\} \subset \mathbb{R}$ and

$$\sum_{i\in\mathbb{N}}\alpha_i b^i$$

is convergent with limit having only a finite number of non-zero components, then $\alpha_i = 0$ for all $i \in \mathbb{N}$. (In particular, this implies each b^i has an infinite number of non-zero components.)

An example of $\{b^i\} \subset l_2$ satisfying (b) is provided by

$$b^i = \frac{1}{i^2}((1-\delta_{i,j})/j^2)_{j\in\mathbb{N}}.$$

Suppose

$$\sum_{i\in\mathbb{N}}\alpha_i b^i = 0.$$

Then, for fixed $j \in \mathbb{N}$, the j-th component is given by

$$\sum_{i\in\mathbb{N}\setminus\{j\}}\alpha_i\frac{1}{j^2 i^2} = 0.$$

Therefore

$$\frac{\alpha_j}{j^2} = \sum_{i\in\mathbb{N}}\frac{\alpha_i}{i^2}$$

and so $\alpha_j = 0$ for all $j \in \mathbb{N}$. If x has only finitely many non-zero components, then

$$P_W x + Af'\big(\sum_{i\in\mathbb{N}}\langle a^i, x\rangle_{l_2}\langle b^i, x\rangle_{l_2}\big)\sum_{i\in\mathbb{N}}a^i\langle b^i, x\rangle_{l_2}$$

has only finitely many non-zero components. If in addition $\nabla V(x)$ has only *finitely many non-zero components, then by* (7.3),

$$\sum_{i \in \mathbb{N}} b^i \langle a^i, x \rangle_{l_2}$$

has only finitely many non-zero components. Hypothesis (b) and (a) then imply $P_p x = 0$. *Now, let* u *be a brake periodic orbit of energy* $h \in \mathbb{R}$ *of*

$$Su''(t) + \nabla V(u) = 0. \tag{7.4}$$

If u *has only finitely many non-zero components then the above argument implies* $P_p u = 0$. *Then equation* (7.4) *and* (7.3) *imply* $(I - P_p)u = 0$. *In conclusion, if* u *is a brake periodic orbit satisfying* (7.4) *and* u *has only finitely many non-zero components, then* $u \equiv 0$. □

Appendix A

Abstract Integration

In this appendix we define and outline the basic properties of the Bochner integral

$$\int_S f(s) \mathrm{d}\mu(s)$$

where f maps from a set S to a Banach space X. Accounts of the Bochner integral in [DunSch58, Yos80] are summarised in the appendix of [Bre73].

Construction of the integral

Let S be a set and μ be a bounded non-negative measure defined on a σ-algebra Σ of S. Now define a semi-norm on the set of functions $f : S \to X$ by

$$|f| = \inf_{\alpha>0} \{\alpha + \mu^*(\{s \in S : \|f(s)\| > \alpha\})\},$$

where, for an arbitrary subset E of S,

$$\mu^*(E) = \inf\{\mu(F) : F \in \Sigma,\ E \subset F\}.$$

A subset E of S is μ-null if $\mu^*(E) = 0$, and a function $f : S \to X$ is a μ-null function if $\{s \in S : \|f(s)\| > \alpha\}$ is μ-null for all $\alpha > 0$. We say that $f = g$ μ-almost everywhere (a.e.) if $f(s) = g(s)$ for all $s \in S \setminus N$ where N is μ-null. A function $f : S \to X$ is a μ-null function if and only if $|f| = 0$ [DunSch58, III.2.4, p.103]. Therefore the relation on the set of functions from S to X given by

$$f \sim g \text{ if and only if } |f - g| = 0$$

is an equivalence relation. The μ-null functions form a linear subspace of the set of functions $f : S \to X$. We denote the equivalence class containing f by $[f]$, and we let $F(S, \Sigma, \mu, X) = \{[f] : f : S \to X\}$ which is a normed linear space with $\|[f]\| = |f|$. We speak of elements of $F(S, \Sigma, \mu, X)$ as functions rather than a set of equivalent functions and we write f in place of $[f]$. Convergence in the normed linear space is called convergence in μ-measure.

A function $f : S \to X$ is said to be μ-measurable if, for every $E \in \Sigma$, $\chi_E f$ lies in the closure of $F(S, \Sigma, \mu; X)$ under $|\cdot|$. The set of μ-measurable functions, denoted $M(S, \Sigma, \mu; X)$, is a closed linear subspace of $F(S, \Sigma, \mu; X)$. If $f : S \to X$ is μ-measurable then $\|f\| : S \to \mathbb{R}$ is μ-measurable [DunSch58, III.2.12, p106].

A function $f : S \to X$ is called a step function if it μ-a.e. equal to a function $g : S \to X$ which has only a finite number of values, and a step function f is said to be μ-integrable if $g^{-1}(\{x\}) \in \Sigma$ for all $x \in X$. For each $E \in \Sigma$ we define

$$\int_E f(s)\,\mathrm{d}\mu(s) = \sum_{x \in X} \mu(E \cap g^{-1}(\{x\}))x,$$

where the sum is finite by the definition of a step function.

A function $f : S \to X$ is said to be μ-integrable if there is a sequence f_n of μ-integrable step functions converging to f in μ-measure with

$$\lim_{m,n \to \infty} \int_S \|f_m(s) - f_n(s)\|\,\mathrm{d}\mu(s) = 0.$$

Then for each $E \in \Sigma$, $\int_E f_n(s)\,\mathrm{d}\mu(s)$ converges in X and the limit is independent of the choice of sequence f_n satisfying the conditions. Finally, for each $E \in \Sigma$, let

$$\int_E f(s)\,\mathrm{d}\mu(s) = \lim_{n \to \infty} \int_E f_n(s)\,\mathrm{d}\mu(s).$$

Denote the set of (equivalence classes of) all μ-integrable functions $f : S \to X$ by $L(S, \Sigma, \mu; X)$, or just $L(S; X)$ if Σ and μ are understood.

Basic properties

The set $L(S,\Sigma,\mu;X)$ is a linear space and, for each $E \in \Sigma$, the map $I_E : L(S,\Sigma,\mu;X) \to X$ defined by

$$I_E(f) = \int_E f(s)\,\mathrm{d}\mu(s)$$

is linear. A μ-measurable function $f : S \to X$ is integrable if and only if the function $\|f\| : S \to \mathbb{R}$ is integrable [DunSch58, III.2.22]. If $f \in L(S,\Sigma,\mu;X)$ then, by [DunSch58, III.2.15 & III.2.16],

$$\|\int_S f(s)\,\mathrm{d}\mu(s)\| \le \int_S \|f(s)\|\,\mathrm{d}\mu(s).$$

Lemma A.1. *Whenever $x^* \in X^*$ and $f \in L(S,\Sigma,\mu;X)$, $x^*(f) \in L(S,\Sigma,\mu;\mathbb{R})$ and*

$$x^* \int_S f(s)\,\mathrm{d}\mu(s) = \int_S x^*(f(s))\,\mathrm{d}\mu(s).$$

Proof. See [DunSch58, III.2.19]. □

Let $1 \le p \le \infty$. Then we denote $L^p(S,\Sigma,\mu;X)$ to be the set of (equivalence classes of) μ-measurable functions $f : S \to X$ such that $\|f\|^p : S \to \mathbb{R}$ is μ-integrable. We endow $L^p(S,\Sigma,\mu;X)$ with the norm

$$\|f\|_p = \left(\int_S \|f(s)\|^p\,\mathrm{d}\mu(s)\right)^{1/p}.$$

Then $L^p(S,\Sigma,\mu;X)$ is a Banach space. If X is a Hilbert space then $L^2(S,\Sigma,\mu;X)$ a Hilbert space with inner product

$$\langle f,g\rangle_p = \int_S \langle f(s),g(s)\rangle_X\,\mathrm{d}s.$$

Theorem A.2. [DunSch58, III.6.16, p151]*(Lebesgue Dominated Convergence) Let $1 \le p < \infty$ and suppose $\{f_n\}$ is a sequence in $L^p(S,\Sigma,\mu;X)$ converging μ-almost everywhere to a function f. Suppose there exists a function $g \in L^p(S,\Sigma,\mu;X)$ such that $|f_n(s)| \le |g(s)|$ μ-almost everywhere. Then*

$$f \in L^p(S,\Sigma,\mu;X) \quad \text{and} \quad \|f_n - f\|_p \to 0 \quad \text{as } n \to \infty.$$

By a theorem of Pettis [Yos80], a function $f : S \to X$ is μ-measurable if and only if

(i) there exists a μ-null set N such that $f(S \setminus N)$ is contained in a separable set, and

(ii) for every $x^* \in X^*$ the function $s \mapsto x^*(f(s))$ on S is μ-measurable.

In our use of the Bochner integral, X is always separable, so condition (i) above is automatically satisfied. Additionally S is a closed bounded interval I in $\mathbb{R}$ with the usual topology. In this case Σ is taken to be the Borel σ-algebra, generated by closed sets in S, μ the corresponding Lebesgue measure. We write $\mathrm{d}\mu(s)$ as $\mathrm{d}s$, $L(S, \Sigma, \mu; X)$ as $L(S; X)$, and we drop reference to μ in μ-measurability and μ-integrability. Since any continuous real-valued function ϕ on S may be uniformly approximated by integrable step functions, ϕ is measurable. So by criteria (i) and (ii) above, any continuous function $f : I \to X$ is measurable and hence, by Theorem A.2, integrable.

Appendix B

Sobolev Spaces

In this appendix we define the space $W^{1,p}(0,1;X)$ and outline some of its properties. The basis of this summary is the appendix of [Bre73] and [ButGia98, Chp. 2].

Definition B.1. *The function $f : [0,1] \to X$ belongs to $W^{1,p}(0,1;X)$ if there exists a function $g \in L^p(0,1;X)$ such that*

$$f(t) = f(0) + \int_0^t g(s)\,\mathrm{d}s \quad \textit{for all } t \in [0,1].$$

Note that if $f \in W^{1,p}(0,1;X)$ then f is absolutely continuous. A point $t \in (0,1)$ is called a Lebesgue point of $g \in L^p(0,1;X)$ if

$$\lim_{h \to 0,\, h \neq 0} \frac{1}{h} \int_t^{t+h} \|g(s) - g(t)\|\,\mathrm{d}t = 0.$$

The set of Lebesgue points of $g \in L^p(0,1;X)$ has full measure [DunSch58, III.12.9] and

$$\lim_{h \to 0,\, h \neq 0} \frac{1}{h} \int_t^{t+h} g(s)\,\mathrm{d}s = g(t) \quad \text{a.e } t \in [0,1].$$

In particular, $f \in W^{1,p}(0,1;X)$ is Fréchet (& Gateaux) differentiable for almost all $t \in [0,1]$ and

$$\mathrm{d}f[t]s = f'(t)s = g(t)s \quad \text{for all } s \in \mathbb{R}.$$

The vector space $W^{1,p}(0,1;X)$ is a Banach space when endowed with the norm given by

$$\|f\|_{W^{1,p}(0,1;X)} = \left(\int_0^1 \|f(s)\|^p + \|f'(s)\|^p \, \mathrm{d}s \right)^{1/p},$$

and when X is a real Hilbert space, $W^{1,2}(0,1;X)$ is a real Hilbert space with inner product

$$\langle f_1, f_2 \rangle_{W^{1,2}(0,1;X)} = \int_0^1 \langle f_1(s), f_2(s) \rangle + \langle f'_1(s), f'_2(s) \rangle \, \mathrm{d}t.$$

Lemma B.2. *Let X be a separable real Hilbert space with basis $\{e_n : n \in \mathbb{N}\}$ and let $f, g \in L^2(0,1;X)$. Suppose*

$$\int_0^1 \langle f(s), \phi'(s) e_n \rangle \, \mathrm{d}s = -\int_0^1 \langle g(s), \phi(s) e_n \rangle \, \mathrm{d}t$$

for all $\phi \in C_0^\infty(0,1;\mathbb{R})$ and all $n \in \mathbb{N}$. Then $f \in W^{1,2}(0,1;X)$ and $f' = g$ almost everywhere.

Proof. Fix $n \in \mathbb{N}$. Then by integration by parts

$$\int_0^1 \left(\langle f(s), e_n \rangle - \int_0^s \langle g(t), e_n \rangle \, \mathrm{d}t \right) \phi'(s) \, \mathrm{d}t = 0$$

for all $\phi \in C_0^\infty(0,1;\mathbb{R})$. By DuBois-Reymond's lemma [ButGia98, Lemma 1.8, p.15] there is a constant $c_n \in \mathbb{R}$ such that

$$\langle f(s), e_n \rangle - \int_0^s \langle g(t), e_n \rangle \, \mathrm{d}t = c_n \quad \text{a.e } s \in [0,1]$$

and so by Lemma A.1 and Riesz's Theorem

$$\langle f(s) - \int_0^s g(t) \, \mathrm{d}t, e_n \rangle = c_n \quad \text{a.e } s \in [0,1]. \tag{B.1}$$

Since a countable union of null sets is null, (B.1) holds for all $n \in \mathbb{N}$ and

$$\langle f(s) - \int_0^s g(t) \, \mathrm{d}t, e_n \rangle = \langle f(0), e_n \rangle$$

for all $n \in \mathbb{N}$. Therefore

$$f(s) - \int_0^s g(t)\,\mathrm{d}t = f(0) \quad \text{a.e } s \in [0,1]$$

Hence $f \in W^{1,2}(0,1;X)$ and $f' = g$ almost everywhere. $\square$

Lemma B.3. *Let X be a Banach space and $q \in W^{1,2}(0,1;X)$. Then $\|q(t)\| \leq 2\|q\|_{W^{1,2}(0,1;X)}$ for all $t \in (0,1)$.*

Proof. By the definition of $W^{1,2}(0,1;X)$, for all $t \in (0,1)$,

$$q(t) = q(0) + \int_0^t q'(\tau)\,\mathrm{d}\tau.$$

Hence for all $s, t \in (0,1)$,

$$\|q(t)\| \leq \|q(s)\| + \|\int_s^t q'(\tau)\,\mathrm{d}\tau\| \leq \|q(s)\| + \int_0^1 \|q'(\tau)\|\,\mathrm{d}\tau.$$

Integrating with respect to s and applying the Cauchy-Schwarz inequality gives

$$\|q(t)\| \leq \left(\int_0^1 \|q(s)\|^2\,\mathrm{d}s\right)^{1/2} + \left(\int_0^1 \|q'(s)\|^2\,\mathrm{d}s\right)^{1/2} \leq 2\|q\|_{W^{1,2}(0,1;X)}$$

as required. $\square$

Appendix C

Regularity of Functionals

Let X be a Hilbert space and let $V \in C^1(X, [0, \infty))$. Consider the functional $\mathcal{V} : W^{1,2}(0,1;X) \to [0, \infty)$ defined by

$$\mathcal{V}(q) = \int_0^1 V(q(t))\,\mathrm{d}t$$

for all $q \in W^{1,2}(0,1;X)$. Note that since elements in $W^{1,2}(0,1;X)$ are continuous, $\mathcal{V}$ is well defined.

Theorem C.1. *Let $V \in C^1(X, [0, \infty))$. Then $\mathcal{V} \in C^1(W^{1,2}(0,1;X), [0, \infty))$ and*

$$\mathrm{d}\mathcal{V}[q]r = \int_0^1 \langle \nabla V(q(t)), r(t) \rangle\,\mathrm{d}t$$

for all $q, r \in W^{1,2}(0,1;X)$.

Proof. We will prove that $\mathcal{V} : W^{1,2}(0,1;X) \to [0, \infty)$ is continuously Gateaux differentiable with

$$\mathcal{V}'(q)r = \int_0^1 \langle \nabla V(q(s)), r(s) \rangle\,\mathrm{d}s \quad \text{for all } q, r \in W^{1,2}(0,1;X).$$

Then we will show that $\mathcal{V}' : W^{1,2}(0,1;X) \to W^{1,2}(0,1;X)^*$ is continuous and so, by [ChoHal82, Theorem 1.3], $\mathcal{V} \in C^1(W^{1,2}(0,1;X), [0,\infty))$ and $\mathcal{V}'(q) = \mathrm{d}\mathcal{V}[q]$ for all $q \in W^{1,2}(0,1;X)$.

Fix q and r in $W^{1,2}(0,1;X)$. Since $V \in C^1(X, [0, \infty))$ the function $t \mapsto V(q(s) + tr(s))$ is differentiable for each $s \in (0,1)$. So, by the Mean Value

Theorem,

$$t^{-1}\{V(q(s)+tr(s))-V(q(s))\}=\langle\nabla V(q(s)+\theta(s,t)r(s)),r(s)\rangle \tag{C.1}$$

where $\theta(s,t)\in(0,t)$. In the following we do not assume that θ is a measurable function of s. Note that the measurability of $s\mapsto\langle\nabla V(q(s)+\theta r(s)),r(s)\rangle$ follows from the equality (C.1). Since $\nabla V : X\to X$ is continuous and $\theta\to0$ as $t\to0$, for each $s\in(0,1)$

$$|\langle\nabla V(q(s)+\theta(s,t)r(s))-\nabla V(q(s)),r(s)\rangle|\to 0 \tag{C.2}$$

as $t\to0$. Since the function $(s,t)\mapsto q(s)+tr(s)$ is a continuous function from $[0,1]\times[0,1]$ into X, set $\{q(s)+\theta r(s) : s\in(0,1)\}$ is contained in a compact subset of X and since ∇V is continuous,

$$\{|\langle\nabla V(q(s)+\theta(s,t)r(s))-\nabla V(q(s)),r(s)\rangle| : s,t\in(0,1)\} \tag{C.3}$$

is bounded in $\mathbb{R}$. The equations (C.2), (C.3) allow us to use the Dominated Convergence Theorem below. Let $t_n>0$ with $t_n\to0$ as $n\to\infty$. Then

$$\begin{aligned}
&\left|\frac{\mathcal{V}(q+t_nh)-\mathcal{V}(q)}{t_n}-\int_0^1\langle\nabla V(q(s)),r(s)\rangle\,\mathrm{d}t\right|\\
&=\left|\int_0^1 t_n^{-1}\{V(q(s)+t_nr(s))-V(q(s))\}-\langle\nabla V(q(s)),r(s)\rangle\,\mathrm{d}s\right|\\
&=\left|\int_0^1\langle\nabla V(q(s)+\theta(s,t_n)r(s))-\nabla V(q(s)),r(s)\rangle\,\mathrm{d}s\right|,\quad\text{by (C.1)},\\
&\le\int_0^1|\langle\nabla V(q(s)+\theta(s,t_n)r(s))-\nabla V(q(s)),r(s)\rangle|\,\mathrm{d}s\to0
\end{aligned}$$

as $n\to\infty$. Hence $\mathcal{V}$ is Gateaux differentiable with

$$\mathcal{V}'(q)r=\int_0^1\langle\nabla V(q(s)),r(s)\rangle\,\mathrm{d}s\quad\text{for all } q,r\in W^{1,2}(0,1;X).$$

We conclude the proof by showing that $\mathcal{V}' : W^{1,2}(0,1;X)^*\to W^{1,2}(0,1;X)^*$ is continuous. Let $q^n \to q$ as $n \to \infty$ in $W^{1,2}(0,1;X)$. Then

$$\begin{aligned}
\|\mathcal{V}'(q^n) - \mathcal{V}'(q)\|_{W^{1,2}(0,1;X)^*} &= \sup_{\|r\|_{W^{1,2}(0,1;X)}=1} |\mathcal{V}'(q^n)r - \mathcal{V}'(q)r| \\
&= \sup_{\|r\|_{W^{1,2}(0,1;X)}=1} \left| \int_0^1 \langle \nabla V(q^n(t)) - \nabla V(q(s)), r(s)\rangle \, \mathrm{d}t \right| \\
&\le \left(\int_0^1 \|\nabla V(q^n(s)) - \nabla V(q(s))\|^2 \, \mathrm{d}t \right)^{1/2}.
\end{aligned}$$

Since $\{q(s) : s \in [0,1]\}$ is compact in X and, by Lemma B.3, $q^n \to q$ uniformly in X, the function

$$s \mapsto \|\nabla V(q^n(s)) - \nabla V(q(s))\|^2$$

is uniformly bounded and point-wise convergent to 0. So the Dominated Convergence Theorem applies to give

$$\|\mathcal{V}'(q^n) - \mathcal{V}'(q)\|_{W^{1,2}(0,1;X)^*} \to 0 \quad \text{as } n \to \infty.$$

$\square$

Appendix D

Brouwer degree

This summary is based on [Llo78] where a fuller account may be found. Let $\mathfrak{D}$ be the set of bounded open subsets of $\mathbb{R}^N$. For $D \in \mathfrak{D}$, denote the boundary of D with respect to $\mathbb{R}^N$ by ∂D and let $C(\overline{D})$ denote the space of continuous functions $f : \overline{D} \to \mathbb{R}^N$ equipped with the supremum norm. A triple (f, D, p) is said to be admissible if

$$D \in \mathfrak{D}, \quad f \in C(\overline{D}) \quad \text{and} \quad p \in \mathbb{R}^N \setminus f(\partial D).$$

For each admissible (f, D, p) there is an associated integer $\mathrm{d}(f, D, p)$, called the Brouwer degree of f at p relative to D, with the following properties:

(i) if $D \in \mathfrak{D}$ and $p \in D$ then $\mathrm{d}(I_{\overline{D}}, D, p) = 1$,

(ii) if $D_1, D_2, D \in \mathfrak{D}$ with $D_1 \cap D_2 = \emptyset$ and $D_1, D_2 \subset D$, and $f \in C(\overline{D})$ with $p \notin \overline{f(\overline{D} \setminus (D_1 \cup D_2))}$ then

$$\mathrm{d}(f, D, p) = \mathrm{d}(f_{|D_1}, D_1, p) + \mathrm{d}(f_{|D_2}, D_2, p),$$

(iii) if $D \in \mathfrak{D}$ and $h : [0, 1] \to C(\overline{D})$ is continuous, then provided $p \notin h(t)(\partial D)$ for all $t \in [0, 1]$ then $\mathrm{d}(h(t), D, p)$ is independent of $t \in [0, 1]$,

(iv) if (f, D, p) is admissible then $(f - p, D, 0)$ is admissible and

$$\mathrm{d}(f, D, p) = \mathrm{d}(f - p, D, 0).$$

If (f, D, p) is admissible with $p \notin f(\overline{D})$, then $\mathrm{d}(f, D, p) = 0$. The converse of this result is very important in applications; if $\mathrm{d}(f, D, p) \neq 0$, then there is an $x \in D$ with $f(x) = p$. In particular, combining this with (iii) gives the following theorem.

Theorem D.1. *Suppose there exists a continuous $h : [0,1] \to C(\overline{D})$ such that $h(0) = I_{|\overline{D}}$ and $(h(t), D, p)$ is admissible for all $t \in [0,1]$. Then if $p \in D$ there exists $x \in D$ such that $h(1)(x) = p$.* $\square$

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